Quality Competition and Multiple Equilibria

Michael H. Riordan*
Department of Economics, Columbia University

January 2009

Abstract

Mixed and pure equilibria are characterized for a duopoly model of simultaneous quality improvement followed by Bertrand price competition with homogeneous consumers. In the second stage Bertrand equilibrium, the firm with highest quality outcome captures the entire market. If firms' initial qualities are not too different, then there are a large number of equilibria of the full game. The equilibria have substantially different expected welfare, and predict a wide range of possible quality outcomes. Finally, it is possible to "purify" mixed equilibria, and multiple pure equilibria exist in extended games of incomplete information.

1. Introduction

Policy makers often presume that competition works well, i.e. restraints on competition are likely to result in higher prices, less product variety, and lower quality. This paper casts doubt on the presumption that unrestrained competition can be counted on to promote efficient investments in product quality. The reason is that quality competition games can have multiple equilibria with very different welfare properties.

The paper models a winner-take-all market in which two firms compete by simultaneously investing in product improvement. The model exhibits multiple equilibria, including mixed equilibria, except when one of the two firms has a sufficiently large initial quality advantage. The mixed equilibria produce inefficiently low quality

* A previous version was presented at a conference in honor of James Friedman at Duke University in 2006 under the title "Quality Competition in a Winner-Take-All-Market". I thank conference participants, Kyle Bagwell, and especially Richard Gilbert for helpful comments and discussions. The paper builds on the model and some results in Gilbert and Riordan (2003, 2007).

investments by the winning firm, and redundant investments by the losing firm. If one firm possesses an initial quality advantage, then equilibria are necessarily discrete, i.e. the supports of equilibrium mixed strategies contain a finite number of quality levels. The equilibria are computed from a pair of difference equations determining the supports.

In the special case of a quadratic cost of quality improvement, it is possible to compute the entire set of equilibria. The number of equilibria is finite and odd in the asymmetric case. If the firms are symmetric *ex ante*, then there is a countable infinity of discrete mixed equilibria. The mixed equilibria typically are less efficient than a pure equilibrium in which the lagging firm invests efficiently in quality improvement and wins the market. A great variety of equilibrium quality outcomes are possible.

The paper is organized as follows. Section 2 presents a duopoly model of a winner-take-all market with endogenous quality. Section 3 characterizes pure equilibria and a symmetric continuous mixed equilibrium for the symmetric case. Section 4 characterizes a necessary local optimality condition for equilibria, and Section 5 uses it to describe discrete mixed equilibria. Section 6 characterizes difference equations for computing the supports of discrete mixed equilibria, and Section 7 computes the equilibria for the case of quadratic costs of quality improvement. Section 7 also characterizes equilibrium welfare, and shows with an example that different equilibria exhibit substantially different welfare, and that equilibrium predictions can cover a wide range of possible quality outcomes. Section 8 shows that the mixed equilibria approximate the pure equilibria by of nearby game of incomplete information. Section 9 concludes by discussing relationships with the theoretical and empirical entry literature. The Appendix contains a detailed proof for the key result in Section 5.

2. Model

Consider the following game. There are two players, indexed $i \in \{1,2\}$. An action for Player *i* is a non-negative real number, $q_i \in R^+$. The payoff of Player 1 is

$$\pi_1(q_1,q_2) = \max\{0, q_1 - q_2 + \Gamma\} - r(q_1),$$

and of Player 2 is

$$\pi_2(q_1,q_2) = \max\{0, q_2 - q_1 - \Gamma\} - r(q_2),$$

where $\Gamma \geq 0$, and $r(q)$ is a differentiable, strictly increasing, convex function with $r(0) = r'(0) = 0$ and $r'(1) = 1$. Thus payoff functions are not quasi-concave. The function π_i has a local maximum at $q_i = 1$, and may have a second local maximum at $q_i = 0$.

The game has the following interpretation. There are two firms competing in a market. The firms first choose the qualities of their products, and then choose prices. Firm 1 is endowed with an initial quality advantage Γ. For each firm, the cost of improving quality by an amount *q* is $r(q)$. The quality investments are endogenous fixed costs. After selecting product qualities, the firms choose prices in Bertrand competition. Consumers are homogeneous, and in equilibrium purchase the higher quality product at a price that reflects the quality difference. The size of the population of consumers is normalized to unity. Production costs are normalized to zero and the profits of Firm *i* are $\pi_i(q_1,q_2)$. These are reduced form profit functions that incorporate the Bertrand equilibrium of the price subgame. The unit of measurement for quality is normalized so that $r'(1) = 1$. Thus $q = 1$ is the "efficient quality" that maximizes joint surplus.

Each firm's quality choice is restricted without loss of generality to the compact interval $Q \equiv [0, \overline{q}]$, where $\overline{q} + \Gamma = r(\overline{q})$. The reason is that any quality greater than $\overline{q}$ is strictly dominated by $q_i = 0$. Note that $\overline{q} > 1$ by construction.

A strategy for Firm *i* is a cumulative probability distribution over quality. Thus a strategy is a function $F_i(q)$ with domain Q and range [0,1]. It is monotone, continuous from the right, and satisfies $F_i(\overline{q}) = 1$. The support of a strategy is

$$S_i = \{q \in Q \mid F_i(q) > 0 \text{ and } F_i(q') < F_i(q) \text{ if } q' < q\}.$$

S_i is closed because $F_i(q)$ is right continuous. Let $\overline{q}_i = \max S_i$ and $\underline{q}_i = \min S_i$.

The expected profit of Firm 1 for quality q_1 is

$$\Pi_1(q_1) = \int_{\underline{q}_2}^{q_1+\Gamma} (q_1 + \Gamma - q)dF_2(q) - r(q_1).$$

Similarly, the expected profit of Firm 2 is

$$\Pi_2(q_2) = \int_{\underline{q}_1}^{q_2-\Gamma} (q_2 - \Gamma - q)dF_1(q) - r(q_2).$$

Equilibrium of the quality game is defined in the usual manner. A pair of strategies (F_1, F_2) form a Nash equilibrium if

$$S_i \subset \arg\max_{q \in Q} \Pi_i(q).$$

All elements in the support of an equilibrium mixed strategy have the same expected profit, i.e. $\Pi_i(q_i) = \Pi_i(\overline{q}_i)$ for $q_i \in S_i$ ("Indifference").

3. Usual Suspects

If the two firms are symmetric, then there exists a symmetric equilibrium in which each firm randomizes over quality improvements ranging from 0 to 1.

Proposition 1 (“Symmetry”): If $\Gamma = 0$, then there is a symmetric equilibrium in which both firms choose the strategy $F(q) = r'(q)$ with the support $S = [0,1]$.

Proof: The symmetric strategy satisfies the necessary equilibrium indifference and optimality conditions, $\int_0^q (q-z)dF(z) - r(q) = 0$ and $F(q) = r'(q)$ for all $q \in S$. Moreover, a firm’s deviation profit is lower for any $q > 1$ by the convexity of $r(q)$.■

A pure equilibrium has the property that both S_i are singleton sets. There always exists a pure equilibrium in which Firm 1 chooses the efficient quality, and Firm 2 declines to invest. There exists a second pure equilibrium that reverses these roles if Firm 1’s initial quality advantage is not too great.

Proposition 2 (“Purity”): There exists a pure equilibrium in which Firm 1 chooses $q_1 = 1$ and Firm 2 chooses $q_2 = 0$. If $r(1) \geq \Gamma$, then there exists a second pure equilibrium in which Firm 1 chooses $q_1 = 0$ and Firm 2 chooses $q_2 = 1$. There are no other pure equilibria.

Proof: Straightforward.

4. Local Optimality

Equilibrium requires each element of the support to maximize expected profit over an interval for which the probability of "winning" is constant.

Proposition 3 ("Local Optimality"): In equilibrium, $F_2(q_1+\Gamma)=r'(q_1)$ for $q_1 \in S_1$, and $F_1(q_2-\Gamma)=r'(q_2)$ for $q_2 \in S_2$.

Proof: This follows from necessary conditions for optimality. Let $F_i^+(q)=\lim_{\tilde{q}\downarrow q} F(\tilde{q})$ and $F_i^-(q)=\lim_{\tilde{q}\uparrow q} F(\tilde{q})$. In equilibrium, a necessary condition for $q_1 = S_1$ is $F_2^+(q_1+\Gamma)\le r'(q_1)\le F_2^-(q_1+\Gamma)$; otherwise, Firm 1 could increase its expected profit by choosing a slightly lower or higher quality. This local optimality condition implies $F_2(q_1+\Gamma)=r'(q_1)$ for $q_1 \in S_1$, because $F_i(q)$ is increasing and $F_i^+(q)\ge F_i(q)\ge F_i^-(q)$. Similarly, $F_1^+(q_2-\Gamma)\le r'(q_2)\le F_1^-(q_2-\Gamma)$ for $q_2 = S_2$ implies $F_1(q_2-\Gamma)=r'(q_2)$ for $q_2 \in S_2$. ■

Local Optimality is important in what follows for two related reasons. First, it links Firm 1's equilibrium mixed strategy to Firm 2's equilibrium support, and *vice versa*. Thus it is possible to construct equilibrium mixed strategies from a knowledge of equilibrium supports. Second, Local Optimality together with Indifference is instrumental for charactering equilibrium supports. Thus a method for describing the equilibria of the game is to first describe equilibrium supports, and second describe corresponding probability distributions.

5. Discreteness

A continuous mixed equilibrium has the property that the S_i are compact intervals. Such an equilibrium exists if the firms are symmetric (Proposition 1). A continuous mixed equilibrium does not exist if the firms are asymmetric. If the firms are asymmetric then equilibrium supports necessarily are discrete sets with the same cardinality. Moreover, there are two types of equilibria depending on which firm is "at the top of the totem pole". Note that the pure equilibria are included as a special case in following description.

<u>Proposition 4 ("Discreteness")</u>: If $\Gamma > 0$, then in equilibrium $S_1 = \{q_1^1, q_1^2, ...q_1^n\}$ and $S_2 = \{q_2^1, q_2^2, ...q_2^n\}$ for a positive integer *n*. Moreover, either

$$q_1^j + \Gamma > q_2^j$$

for $j = 1,...n$, with $q_1^n = 1$ and $q_2^1 = 0$ ("Type A equilibrium"); or

$$q_2^j > q_1^j + \Gamma$$

for $j = 1,...n$, with $q_1^0 = 0$ and $q_2^n = 1$ ("Type B equilibrium ").

<u>Proof</u>: See Appendix.

This proposition is a key result. With knowledge of equilibrium supports in hand, it is straightforward to use Local Optimality to construct the corresponding equilibrium strategies for the two types of equilibria. The following result follows from Discreteness and Local Optimality.

Corollary: Assume $\Gamma > 0$.

(A) In Type A equilibrium: $F_1(q_1^k) = r'(q_2^{k+1})$ for $k = 1,...n-1$, and $F_1(q_1^n) = 1$; and $F_2(q_2^k) = r'(q_1^k)$ for $k = 1,...n$.

(B) In Type B equilibrium: $F_1(q_1^k) = r'(q_2^k)$ for $k = 1,...n$; and $F_2(q_2^k) = r'(q_1^{k+1})$ for $k = 1,...n-1$, and $F_2(q_2^n) = 1$.

6. Computation

All elements are in place to compute both types of equilibria.

A. Type A Equilibria

Suppose there exists a Type A mixed equilibrium ($n \geq 2$). Let α_i^j denote the probability that Firm *i* selects $q_i^j \in S_i$, i.e.

$$\alpha_i^1 = F_i(q_i^1)$$

and

$$\alpha_i^j = F_i(q_i^j) - F_i(q_i^{j-1})$$

for $j = 2,..n$. Using this notation, the expected profit of Firm 1 is

$$\Pi_1(q_1^k) = \sum_{j=1}^{k} \alpha_2^j (q_1^k + \Gamma - q_2^j) - r(q_1^k)$$

and for $k = 1,...n$. Local Optimality requires $r'(q_1^k) = \sum_{j=1}^{k} \alpha^j = F_2(q_2^k)$. Therefore, for $k = 1,...n$,

$$\Pi_1(q_1^k) = \sum_{j=1}^{k} [r'(q_1^j) - r'(q_1^{j-1})](q_1^k + \Gamma - q_2^j) - r(q_1^k)$$

with $q_1^0 \equiv 0$. In equilibrium, there is number $\Pi > 0$ such that $\Pi_1(q_1^k) = \Pi$ for $q_1^k \in S_1$. The reason why Firm 1 expects strictly positive profit in equilibrium is that $\alpha_2^1 > 0$ implies that Firm 1 earns an expected profit of at least $\alpha_2^1 \Gamma > 0$. Given S_2, it is possible to solve for S_1 by imposing this indifference condition. The precise value of Π is pinned down by the boundary condition $q_1^n = 1$.

Alternatively, the equilibrium conditions for Firm 1 imply $q_1^n = 1$, and

$$\Pi_1(q_1^k) - \Pi_1(q_1^{k-1}) = \alpha_2^k [q_1^k + \Gamma - q_2^k] + \sum_{j=1}^{k-1} \alpha_2^j [q_1^k - q_1^{k-1}] - r(q_1^k) + r(q_1^{k-1})$$

for $k = 2,...n$. Therefore, Indifference and Local Optimality further imply

$$[r'(q_1^k) - r'(q_1^{k-1})][q_1^k + \Gamma - q_2^k] + r'(q_1^{k-1})[q_1^k - q_1^{k-1}] = r(q_1^k) - r(q_1^{k-1}),$$

or, equivalently,

$$r'(q_1^k)[q_1^k + \Gamma - q_2^k] + r'(q_1^{k-1})[q_2^k - q_1^{k-1} - \Gamma] = r(q_1^k) - r(q_1^{k-1})$$

for $k = 2,...n$. Given $S_2 = \{q_2^1,...q_2^n\}$, this is a first-order difference equation in q_1^k. Starting with $q_1^n = 1$, the equation can be solved recursively for values of q_1^{k-1}. For example, q_1^{n-1} solves

$$[1+\Gamma - q_2^n] + r'(q_1^{n-1})[q_2^n - q_1^{n-1} - \Gamma] = r(1) - r(q_1^{n-1}) \ ,$$

Moreover, Firm 1 has no incentive unilaterally to deviate from this computed strategy because, given the convexity of $r(q)$, q_1^1 is by construction optimal on the interval $[0, q_2^2 - \Gamma)$, q_1^k is optimal on $[q_2^k - \Gamma, q_2^{k+1} - \Gamma)$ for $k = 2,..n-1$, and $q_1^n = 1$ is optimal on $[q_2^n - \Gamma, \overline{q}]$.

Similarly, still supposing a Type A equilibrium, it is possible to solve for S_2 given S_1 by imposing the break-even condition $\Pi_2(q_2^k) = \Pi_2(0) = 0$ and the boundary condition $q_2^1 = 0$. We have

$$\Pi_2(q_2^k) = \sum_{j=1}^{k-1} \alpha_1^j (q_2^k - \Gamma - q_1^j) - r(q_2^k) \ .$$

Therefore, $\Pi_2(q_1^k) - \Pi_2(q_1^{k-1}) = 0$ implies (for $k = 2,...n$)

$$\alpha_1^{k-1}(q_2^k - \Gamma - q_1^{k-1}) + \sum_{j=1}^{k-2} \alpha_1^j (q_2^k - q_2^{k-1}) = r(q_2^k) - r(q_2^{k-1}) \ .$$

Local optimality then implies

$$[r'(q_2^k) - r'(q_2^{k-1})](q_2^k - \Gamma - q_1^{k-1}) + r'(q_2^{k-1})(q_2^k - q_2^{k-2}) = r(q_2^k) - r(q_2^{k-1}) \ ,$$

or, equivalently,

$$r'(q_2^k)(q_2^k - \Gamma - q_1^{k-1}) + r'(q_2^{k-1})(q_1^{k-1} + \Gamma - q_2^{k-1}) = r(q_2^k) - r(q_2^{k-1})$$

Given S_1 and $q_2^1 = 0$, this equation is solved recursively for q_2^k. Firm 2 has no incentive deviate from this computed strategy by the convexity of $r(q)$.

Summarizing, the computation a candidate Type A mixed equilibrium amounts to using Indifference to find a find point in (S_1, S_2). The candidate must also satisfy

Discreteness. These conditions and Local Optimality are together sufficient for the existence of equilibrium.[1]

Proposition 5A ("Computation"): There exists a Type A equilibrium with $S_1 = \left\{q_1^1, \ldots q_1^n\right\}$ and $S_2 = \{q_2^1, \ldots q_2^n\}$, and if and only if: $q_1^n = 1$ and

$$r'(q_1^k)[q_1^k + \Gamma - q_2^k] + r'(q_1^{k-1})[q_2^k - q_1^{k-1} - \Gamma] = r(q_1^k) - r(q_1^{k-1})$$

for $k = 2, \ldots n$; and $q_2^1 = 0$ and

$$r'(q_2^k)(q_2^k - \Gamma - q_1^{k-1}) + r'(q_2^{k-1})(q_1^{k-1} + \Gamma - q_2^{k-1}) = r(q_2^k) - r(q_2^{k-1})$$

for $k = 2, \ldots n$.

Implicit in the statement of the proposition are the requirements of Discreteness for a Type A equilibrium. In particular, $q_1^j + \Gamma > q_2^j$ for $j = 1, \ldots n$, with $q_1^n = 1$ and $q_2^1 = 0$.

B. Type B Equilibria

The computation of a Type B equilibrium is similar, except that there is an additional condition that must be satisfied for sufficiency. This No-Leapfrogging condition assures that the more efficient Firm 1 does not have an incentive to choose $q_1 = 1$ and win with probability one.

---

[1] By Local Optimality, no deviation from S_1 is profitable for Firm 1, and do deviation from S_2 belonging to $[0, \overline{q}]$ is profitable for Firm 2. Furthermore, assuming $\overline{q} > 1 + \Gamma$, $q_2 = 1 + \Gamma$ is the most profitable deviation in $[1 + \Gamma, \overline{q}]$ for Firm 2 by convexity of $r(\bullet)$.

Proposition 5B ("Computation ", cont'd): There exists a Type B equilibrium, with

$S_1 = \left\{ q_1^1, \ldots q_1^n \right\}$ and $S_2 = \{ q_2^1, \ldots q_2^n \}$, if and only: $q_2^n = 1$ and

$$r'(q_2^k)[q_2^k - \Gamma - q_1^k] + r'(q_2^{k-1})[q_1^k - q_2^{k-1} + \Gamma] = r(q_2^k) - r(q_2^{k-1})$$

for $k = 2, \ldots n$; $q_1^1 = 0$ and

$$r'(q_1^k)(q_1^k + \Gamma - q_2^{k-1}) + r'(q_1^{k-1})(q_2^{k-1} - \Gamma - q_1^{k-1}) = r(q_1^k) - r(q_1^{k-1})$$

for $k = 2, \ldots n$; and ("No Leapfrogging")

$$\Gamma + \sum_{k=2}^{n} (q_2^k - q_2^{k-1}) r'(q_1^n) \leq r(1) .$$

No Leapfrogging assures that Firm 1 has no incentive to deviate with $q_1 = 1$, which most profitable deviation that guarantees victory.

7. Quadratic Case

A. Type A Equilibria

The quadratic case has $r(q) = \frac{1}{2} q^2$. Applying Proposition 5A to the quadratic case, a Type A mixed equilibrium with *n* points of support must satisfy the following conditions: $q_1^n = 1$ and

$$q_1^k (q_1^k + \Gamma - q_2^k) + q_1^{k-1}(q_2^k - q_1^{k-1} - \Gamma) = \frac{1}{2}(q_1^k)^2 - \frac{1}{2}(q_1^{k-1})^2$$

for $k = 2, \ldots n$; and $q_2^1 = 0$ and

$$q_2^k (q_2^k - \Gamma - q_1^{k-1}) + q_2^{k-1}(q_1^{k-1} + \Gamma - q_2^{k-1}) = \frac{1}{2}(q_2^k)^2 - \frac{1}{2}(q_2^{k-1})^2$$

for $k = 2, \ldots n$. These difference equations simplify to

$$q_1^{k-1} + \Gamma = \frac{1}{2}\left(q_2^k + q_2^{k-1} \right)$$

and

$$q_2^k - \Gamma = \tfrac{1}{2}\left(q_1^k + q_1^{k-1}\right)$$

for $k = 2,\ldots n$. In addition, each $\{q_i^1,\ldots q_i^n\}$ must be an increasing sequence, and

$$q_1^k + \Gamma > q_2^k$$

for $k = 1,\ldots n$. Implicit are the requirements that $q_2^n < 1+\Gamma$ and $q_1^1 \geq 0$ at the boundaries of the solution to the pair of difference equations.

The above difference equations can be manipulated to imply a simple second-order linear difference equation for Firm 2:

$$q_2^{k+1} - 2q_2^k + q_2^{k-1} = 0$$

for $k = 2,\ldots n-1$. There are two boundary conditions:

$$q_2^1 = 0$$

and

$$q_2^n = \tfrac{2}{3}\left(1+\Gamma\right) + \tfrac{1}{3} q_2^{n-1}.$$

Solving recursively, and using $q_2^1 = 0$, gives $q_2^k = (k-1)q_2^2$. Therefore, the second boundary condition becomes

$$q_2^2 = \frac{2(1+\Gamma)}{2n-1},$$

and the solution for $k = 3,\ldots n$ is

$$q_2^k = (k-1)\left[\frac{2(1+\Gamma)}{2n-1}\right].$$

Obviously this is an increasing sequence. Moreover,

$$q_2^n = (n-1)\left[\frac{2(1+\Gamma)}{2n-1}\right],$$

satisfies the requirement $q_2^n < 1+\Gamma$ for all $n \geq 2$.

Applying these results, $q_1^k + \Gamma = \tfrac{1}{2}\left(q_2^{k+1} + q_2^k\right)$ for $k = 1,\ldots n-1$ implies

$$q_1^1 = \frac{1+\Gamma}{2n-1} - \Gamma.$$

Therefore, $q_1^1 \geq 0$ requires

$$n \leq 1 + \frac{1}{2\Gamma},$$

i.e. for $\Gamma > 0$, *n* must be sufficiently small. As $\Gamma \to \infty$, however, there is a countable infinity of mixed equilibria.

To complete the computation of equilibria and summarize, Type A equilibrium supports are

$$q_1^k = \tfrac{2k-1}{2n-1}\left(1+\Gamma\right) - \Gamma$$

and

$$q_2^k = \tfrac{2(k-1)}{2n-1}(1+\Gamma)$$

for $k = 1,...n$. Using Local Optimality, the cumulative probabilities with which these qualities are chosen are

$$F_1(q_1^k) = \begin{cases} \tfrac{2k}{2n-1}\left(1+\Gamma\right) & \text{for } k = 1,...n-1 \\ 1 & \text{for } k = n \end{cases}$$

and

$$F_2(q_2^k) = \tfrac{2k-1}{2n-1}(1+\Gamma) - \Gamma$$

for $k = 1,...n$, and the corresponding equilibrium frequencies are

$$\alpha_1^k = \begin{cases} \tfrac{2}{2n-1}(1+\Gamma) & \text{for } k = 1,...n-1 \\ 1 - \tfrac{2(n-1)}{2n-1}\left(1+\Gamma\right) & \text{for } k = n \end{cases}$$

and

$$\alpha_2^k = \begin{cases} \tfrac{1}{2n-1}(1+\Gamma) - \Gamma & k = 1 \\ \tfrac{2}{2n-1}(1+\Gamma) & \text{k=2,...n} \end{cases}$$

for $k = 1,...n$. This equilibrium exists if

$$n \le 1 + \frac{1}{2\Gamma}.$$

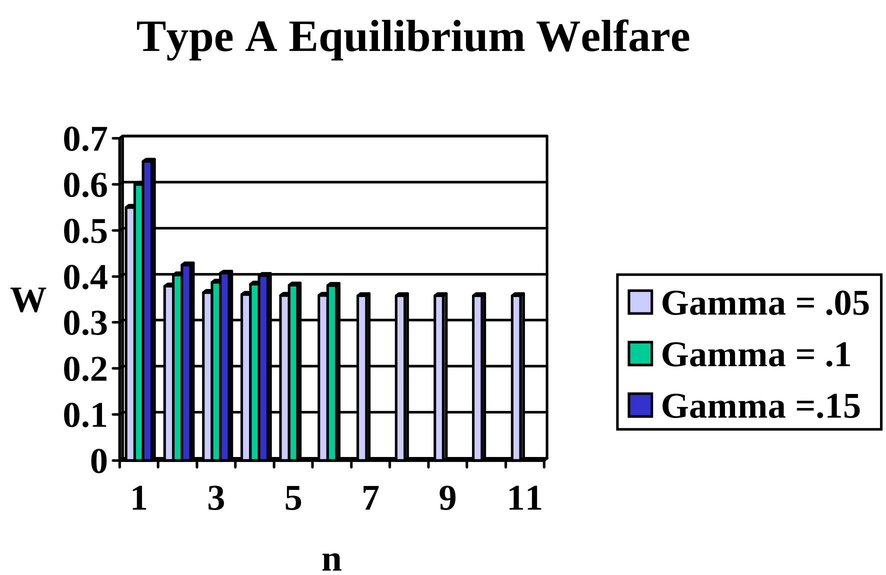
The different possible equilibria have very different welfare properties. The welfare generated by Firm 1 is

$$W_1 = \sum_{k=1}^{n} \alpha_1^k \left[F_2(q_2^k)(q_1^k + \Gamma) - r(q_1^k) \right],$$

and by Firm 2 is

$$W_2 = \sum_{k=1}^{n} \alpha_2^k \left[F_1(q_1^{k-1}) q_2^k - r(q_2^k) \right].$$

The chart below calculates total welfare ($W = W_1 + W_2$) for the quadratic case for selected values of Γ. Note that the number of equilibria varies inversely with Γ. In general, pure equilibria deliver substantially greater welfare.



B. Type B Equilibria

Applying Proposition 5B to the quadratic case gives the conditions: $q_2^n = 1$ and

$$q_2^k(q_2^k - \Gamma - q_1^k) + q_2^{k-1}(q_1^k - q_2^{k-1} + \Gamma) = \tfrac{1}{2}(q_2^k)^2 - \tfrac{1}{2}(q_2^{k-1})^2$$

for $k = 2,...n$; and $q_1^1 = 0$ and

$$q_1^k(q_1^k + \Gamma - q_2^{k-1}) + q_1^{k-1}(q_2^{k-1} - \Gamma - q_1^{k-1}) = \tfrac{1}{2}(q_1^k)^2 - \tfrac{1}{2}(q_1^{k-1})^2$$

for $k = 2,...n$. These difference equations simplify to

$$q_1^{k-1} + \Gamma = \tfrac{1}{2}\left(q_2^k + q_2^{k-1}\right)$$

and

$$q_2^k - \Gamma = \tfrac{1}{2}\left(q_1^k + q_1^{k-1}\right)$$

for $k = 2,...n$. Solving these difference equations gives:

$$q_1^k = \left[\frac{2(k-1)}{2n-1}\right](1-\Gamma)$$

for $k = 1,...n$; and

$$q_2^k = \left[\frac{2k-1}{2n-1}\right]\left(1-\Gamma\right) + \Gamma$$

for $k = 1,...n$. Notice that the solution automatically satisfies the necessary conditions $q_2^k - \Gamma > q_1^k$.

Additionally, the solution must satisfy No-Leapfrogging. In the quadratic case, this condition is

$$\Gamma + \sum_{k=2}^{n}(q_2^k - q_2^{k-1})q_1^k \le \tfrac{1}{2}.$$

Substituting the solution into the difference equations, No Leapfrogging requires:

$$\Gamma + (1-\Gamma)^2\left[\tfrac{4}{(2n-1)^2}\sum_{k=1}^{n-1} k\right] \le \tfrac{1}{2},$$

which, simplifying further, is equivalent to

$$2\Gamma + (1-\Gamma)^2\left[1 - \tfrac{1}{(2n-1)^2}\right] \le 1$$

or

$$n \le \frac{1}{2\Gamma}.$$

There exist exactly one fewer Type B equilibria than Type A equilibria. Thus the number of equilibria when $\Gamma > 0$ is finite and odd.

The cumulative probabilities with which quality improvements are chosen are

$$F_2(q_2^k) = \begin{cases} \frac{2k}{2n-1}\left(1-\Gamma\right) & \text{for } k = 1,\ldots n-1 \\ 1 & \text{for } k = n \end{cases}$$

and

$$F_1(q_1^k) = \frac{2k-1}{2n-1}(1-\Gamma) + \Gamma$$

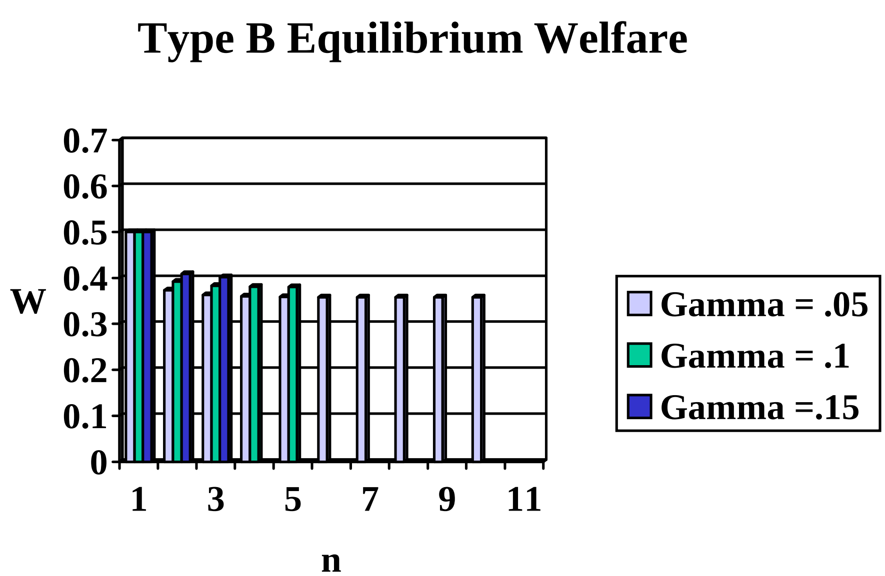
for $k = 1,\ldots n$, and the corresponding equilibrium frequencies are

$$\alpha_2^k = \begin{cases} \frac{2}{2n-1}(1-\Gamma) & \text{for } k = 1,\ldots n-1 \\ 1-\frac{2(n-1)}{2n-1}\left(1-\Gamma\right) & \text{for } k = n \end{cases}$$

and

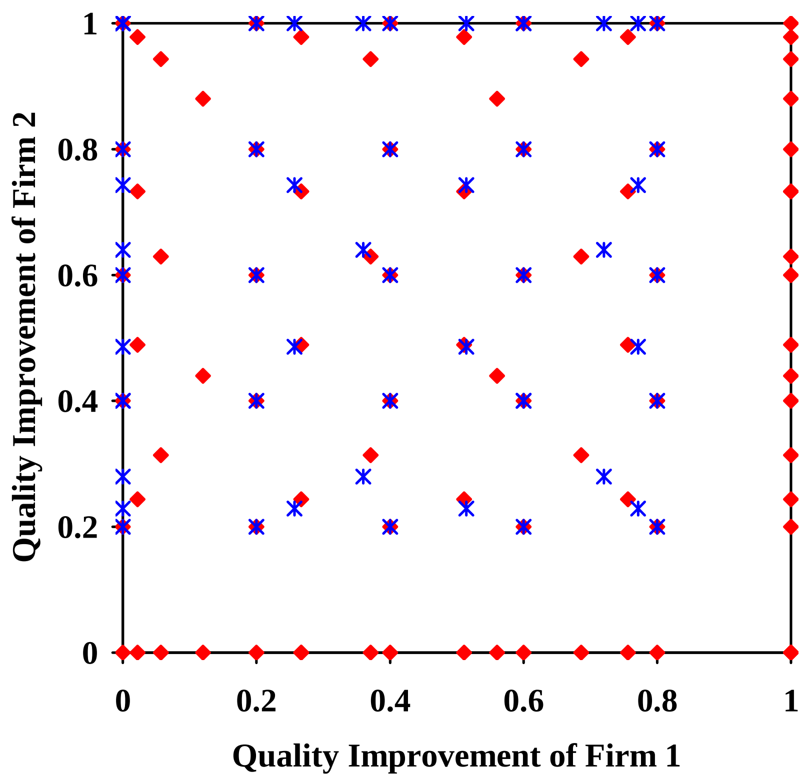
$$\alpha_1^k = \begin{cases} \frac{1}{2n-1}(1-\Gamma) + \Gamma & k = 1 \\ \frac{2}{2n-1}(1-\Gamma) & \text{k=2,...n} \end{cases}$$

for $k = 1,\ldots n$. Welfare is calculated in the chart below.



C. Empirical Implications

Almost anything is possible. The following is a scatter plot of possible equilibrium outcomes for $\Gamma = .1$:



In this case, there are six Type A and five Type B equilibria. Interestingly, there is an interior region of quality improvements at the bottom and right not observed in equilibrium. This region vanishes as $\Gamma \to 0$.

8. Purification

The main properties of the winner-take-all quality competition model are robust to introducing a small amount of incomplete information, and its mixed equilibria are limits of outcome distributions of pure equilibria of an appropriate sequence of incomplete information models. The general idea that a mixed equilibrium can be "purified" in this way stems from Harsanyi (1973). The specific purification used here is patterned after Bagwell and Wolinsky (2002).

Consider for simplicity the symmetric quadratic model, and extend it to an incomplete information model as follows. The cost of quality improvement for Firm *i* in the extended model is $r(q,t_i) = \delta c(t_i)q + \frac{1}{2}q^2$, where $c(t)$ is a continuously decreasing function on $[0,1]$ with $c(0)=1$ and $c(1)=0$, and δ is a small positive constant. Firm *i*'s type t_i is an independent draw from a standard uniform distribution, and each firm privately learns its type before choosing quality. In other respects, quality competition is the same as before.

Bayes-Nash equilibria of the extended model with δ sufficiently small are constructed as follows. For any positive integer *n*, let $\{s_1^1,...s_1^n\}$ and $\{s_2^1,...s_2^n\}$ be two strictly increasing sequences with $s_2^n = 1$, $s_1^1 = 0$, and $s_2^k > s_1^k$, and define

$$\hat{q}_1^k(t) = s_2^k - \delta c(t) \quad \text{for } k \in \{1,...n\}$$

$$\hat{q}_2^k(t) = s_1^k - \delta c(t) \quad \text{for } k \in \{2...n\}$$

Using these functions, and setting $s_2^0 \equiv 0$ and $\hat{q}_2^1 \equiv 0$, further define

$$\hat{\pi}_1^k(t_1) = \tfrac{1}{2}\hat{q}_1^k(t_1)^2 - \sum_{j=1}^{k}\int_{s_2^{j-1}}^{s_2^j} \hat{q}_2^j(t)dt \quad \text{for } k \in \{1,...n\}$$

$$\hat{\pi}_2^k(t_2) = \begin{cases} 0 & \text{for } k=1 \\ \tfrac{1}{2}\hat{q}_2^k(t_2)^2 - \sum_{j=2}^{k}\int_{s_1^{j-1}}^{s_1^j} \hat{q}_1^{j-1}(t)dt & \text{for } k \in \{2,...n\} \end{cases}$$

and require

$$\hat{\pi}_1^k(s_1^k) = \hat{\pi}_1^{k-1}(s_1^k) \qquad \text{for } k \in \{2, \dots n\}$$

$$\hat{\pi}_2^2(s_2^1) = 0$$

$$\hat{\pi}_2^{k+1}(s_2^k) = \hat{\pi}_2^k(s_2^k) \qquad \text{for } k \in \{2, \dots n-1\}$$

Consider the following are equilibrium strategies:

$$q_1(t) = \begin{cases} \hat{q}_1^k(t) & \text{if} \quad s_1^{k+1} > t \geq s_1^k \text{ and } k \in \{1, \dots n-1\} \\ \hat{q}_1^n(t) & \text{if} \quad t \geq s_1^n \end{cases}$$

$$q_2(t) = \begin{cases} 0 & \text{if} \quad s_2^1 \geq t \\ \hat{q}_2^k(t) & \text{if} \quad s_2^k \geq t > s_2^{k-1} \text{ and } k \in \{2, \dots n\} \end{cases}$$

The distribution of outcomes resulting from these strategies converges to the Type A equilibrium corresponding to n of the original game as $\delta \to 0$. It follows from $\hat{q}_1^k(t) \to s_2^k$ and $\hat{q}_2^k(t) \to s_1^k$ that

$$\hat{\pi}_1^k(t_1) \to \tfrac{1}{2}\left(s_2^k\right)^2 - \sum_{j=1}^{k}\left(s_2^j - s_2^{j-1}\right)s_1^j \qquad \text{for } k \in \{1, \dots n\}$$

and

$$\hat{\pi}_2^k(t) \to \tfrac{1}{2}\left(s_1^k\right)^2 - \sum_{j=2}^{k}\left(s_1^j - s_1^{j-1}\right)s_2^{j-1} \qquad \text{for } k \in \{2, \dots n\}$$

Therefore,

$$\tfrac{1}{2}\left(s_2^k\right)^2 - \sum_{j=1}^{k}\left(s_2^j - s_2^{j-1}\right)s_1^k = \tfrac{1}{2}\left(s_2^{k-1}\right)^2 - \sum_{j=1}^{k-1}\left(s_2^j - s_2^{j-1}\right)s_1^{k-1}$$

$$\tfrac{1}{2}\left(s_1^k\right)^2 - \sum_{j=2}^{k}\left(s_1^j - s_1^{j-1}\right)s_2^{j-1} = 0$$

in the limit for $k \in \{2,...n\}$. This system of difference equation is solved by

$s_1^k = \frac{2(k-1)}{2n-1}$ and $s_2^k = \frac{2k-1}{2n-1}$. Therefore,

$$\hat{q}_1^k \to \frac{2k-1}{2n-1}$$

$$\hat{q}_2^k \to \frac{2(k-1)}{2n-1}$$

Furthermore, the probabilities of these outcomes are the same as for the corresponding mixed strategy equilibrium.

It remains to argue that $q_1(t)$ and $q_2(t)$ are equilibrium strategies of the extended game for δ sufficiently small. First, there exist $\{s_1^1,...s_1^n\}$ and $\{s_2^1,...s_2^n\}$ satisfying the requisite conditions.[i] Second, Firm 1's profit function if Firm 2 follows $q_2(t)$ is

$$\pi_1(q,t_1) = s_2^1 q + \sum_{k=2}^{n} \int_{s_2^{k-1}}^{s_2^k} \max\{q - \hat{q}_2^k(t), 0\} dt - r(q,t_1)$$

The $\hat{q}_1^k(t_1)$ are the local maxima of this function, and $\hat{\pi}_1^k(t_1)$ are the corresponding profit levels. Third, by construction, $\hat{\pi}_1^k(s_1^k) = \hat{\pi}_1^{k-1}(s_1^k)$ and $\frac{d\hat{\pi}_1^k(t)}{dt} > \frac{d\hat{\pi}_1^{k-1}(t)}{dt}$. It follows that $q_1(t)$ is a best response. A similar argument establishes that $q_2(t)$ is a best response for Firm 2.

In conclusion, the mixed equilibria of the original game approximate pure equilibria of nearby games of incomplete information. More precisely, the mixed strategies approximate equilibrium beliefs about rivals' incentives to invest in quality improvement. Thus multiplicity of equilibria in quality competition games does not hinge on randomized actions.

9. Conclusion

Simultaneous entry models in which firms make a simple in-or-out decision are known to have multiple equilibria with different welfare consequences (Cabral, 2004; Dixit and Shapiro, 1986; Vettas, 2000). The model presented here can be interpreted as a richer entry model in which firms choose quality as part of the entry decision, i.e. quality is an endogenous sunk cost of entry (Sutton, 1991). The result is that equilibria proliferate with the richer action space if firms are not too different initially.

Many models of vertical product differentiation assume heterogeneous consumer preferences for quality and characterize a pure equilibrium in a two stage game with simultaneous quality choice followed by Bertrand-Nash competition (e.g. Choi and Shin, 1992; Motta, 1993, Shaked and Sutton, 1982, Tirole, 1988; Wauthy, 1996). There is no reason to think, however, that multiple equilibria do not plague heterogeneous consumer price-setting models, as they do the homogeneous consumer model studied here. Indeed, vertical product games differentiation have a similar winner-take-all character if the degree of consumer heterogeneity is small (Tirole, 1988).

Empirical entry models typically postulate a simple in-or-out action set. Attention usually is restricted to pure strategies, allowing inference about underlying profit functions from the number of firms entering the market (e.g. Bresnahan and Reiss, 1991; Berry, 1992). Recent research allows for mixed strategies, trying to draw inferences about equilibrium selection probabilities from the empirical frequency of different entry outcomes (Berry and Tamer, 2006). Symmetric in-or-out entry games typically have only one mixed strategy equilibrium. The inference problem clearly becomes more difficult when endogenous quality substantially increases the number of mixed equilibria under consideration. With sufficient data, it seems possible to draw inferences about equilibrium selection probabilities from the empirical frequency of quality outcomes. In finite samples, inference seems likely to be difficult if there are a large number of equilibria.

Finally, multiple equilibria create problems for economic policy, because of the resulting difficult of predicting policy outcomes. Economic regulation might play a role of eliminating unwanted equilibria. For example, access regulation can influence the number of equilibria (Gilbert and Riordan, 2003). Furthermore, it may be hazardous to base policy on a particular selection rule.

References

Bagwell, Kyle and Asher Wolinsky (2002), "Game Theory and Industrial Organization," in Robert Aumann and Sergiu Harts (eds.), Handbook of Game Theory, Volume 3, North-Holland.

Berry, Steven (1992), "Estimation of a Model of Entry in the Airline Industry", Econometrica 60, 889-917.

__________ and Elie Tamer (2006), "Identification in Models of Oligopoly Entry", invited lecture at the 2005 World Congress of the Econometric Society.

Bresnahan, Timothy and Peter Reiss (1991), "Entry and Competition in Concentrated Markets", Journal of Political Economy 99, 7977-1009.

Cabral, Luis (2004), "Simultaneous Entry and Welfare", European Economic Review 48, 943-957.

Choi, Chong Ju, and Hyun Song Shin (1992), "A Comment on a Model of Product Differentiation", Journal of Industrial Economics 40, 229-231.

Dixit, Avinash and Carl Shapiro (1986), "Entry Dynamics with Mixed Strategies", in L.G. Thomas III (ed.), The Economics of Strategic Planning, Lexington, MA: Lexington Books.

Gilbert, Richard and Michael Riordan (2003), “Product Improvement and Technological Tying in a Winner-Take-All-Market”, Department of Economics Discussion Paper 0304-11, Columbia University.

__________ (2007). “Product Improvement and Technological Tying in a Winner-Take-All-Market”, Journal of Industrial Economics, 55, 113-139.

Harsanyi, John (1973), “Games with Randomly Disturbed Payoffs: A New Rationale for Mixed Strategy Equilibrium Points,” International Journal of Game Theory 2, 1-23.

Motta, Massimo (1993), “Endogenous Quality Choice: Price vs. Quantity Competition”, Journal of Industrial Economics 41, 113-131.

Shaked, Avner and John Sutton (1982), “Relaxing Price Competition Through Product Differentiation”, Review of Economic Studies 49, 3-13.

Sutton, John (1991), Sunk Costs and Market Structure: Price Competition, Advertising, and the Evolution of Concentration, Cambridge MA: MIT Press.

Tirole, Jean (1988), The Theory of Industrial Organization, Cambridge, MA: MIT Press.

Vettas, Nikolaos (2000), “On Entry, Exit, and Coordination with Mixed Strategies”, European Economic Review 44, 1557-1576.

Wauthy, Xavier (1996), “Quality Choice in Models of Vertical Differentiation”, Journal of Industrial Economics 44, 345-353.

Appendix

Proof of Proposition 4

Assume $\Gamma > 0$. The proof proceeds by establishing a series of lemmas. The first three lemmas establish that in equilibrium S_1 and S_2 are discrete sets.

Lemma A1: Either $\bar{q}_1$ is a discrete element of S_1, or $\bar{q}_2$ is a discrete element of S_2.

Proof: Suppose to the contrary that there exists $q' < \bar{q}_1$ and $q'' < \bar{q}_2$ such that $[q', \bar{q}_1] \subseteq S_1$ and $[q'', \bar{q}_2] \subseteq S_2$.

Firm 1's indifference condition requires $F_2(q_2) = r'(q_2 - \Gamma)$ for $q_2 \in [q' + \Gamma, \bar{q}_1 + \Gamma) \subset S_2$. Moreover, $\bar{q}_1 + \Gamma \in S_2$ because S_2 is closed. Therefore, $\bar{q}_1 + \Gamma \leq \bar{q}_2$ by the definition of $\bar{q}_2$.

Similarly, Firm 2' indifference condition requires $F_1(q_1) = r'(q_1 + \Gamma)$ for $q_1 \in [q'' - \Gamma, \bar{q}_2 - \Gamma) \subset S_1$, and $\bar{q}_2 - \Gamma \in S_1$. Therefore, $\bar{q}_1 + \Gamma \geq \bar{q}_2$.

To summarize, if $[q', \bar{q}_1] \subseteq S_1$ and $[q'', \bar{q}_2] \subseteq S_2$, for $q' < \bar{q}_1$ and $q'' < \bar{q}_2$, then $\bar{q}_1 + \Gamma = \bar{q}_2$. Furthermore, $F_2(\bar{q}_2) = F_2(\bar{q}_1 + \Gamma) \geq r'(\bar{q}_1)$ and $F_1(\bar{q}_1) = F_1(\bar{q}_2 - \Gamma) \geq r'(\bar{q}_2)$.

It must also be that $\overline{q}_2 = q*$. Clearly $\overline{q}_2 \geq q*$, for otherwise Firm 2 could more profitably choose $q_2 = q*$. But then, $F_1(\overline{q}_1) \geq r'(\overline{q}_2) \geq r'(q*) = 1$ implies $\overline{q}_2 = q*$ and $\overline{q}_1 = q* - \Gamma$. But then Firm could increase its profit by choosing $q_1 = q*$, a contradiction.

■

Lemma A2: If $\overline{q}_1$ is a discrete element of S_1, then $\overline{q}_2$ is a discrete element of S_2. The converse is also true.

Proof: Suppose to the contrary that $\overline{q}_1$ is a discrete element of S_1, and $[q'', \overline{q}_2] \subseteq S_2$ for $q'' < \overline{q}_2$.

Firm 2's indifference condition requires $F_1(q_1) = r'(q_1 + \Gamma)$ for $q_1 \in [q'' - \Gamma, \overline{q}_2 - \Gamma)$. Thus $F_1(\overline{q}_1) > F_1(\overline{q}_2 - \Gamma) = r'(\overline{q}_1 + \Gamma)$.

Finally, local optimality requires $\overline{q}_1 = q*$ and $F_1(\overline{q}_1) > r'(\overline{q}_1 + \Gamma) = r'(q* + \Gamma) > 1$, a contradiction. Therefore, if $\overline{q}_1$ is discrete, then so is $\overline{q}_2$.

The converse follows from a similar proof by contradiction. Suppose that $\overline{q}_2$ is a discrete element of S_2, and $[q', \overline{q}_1] \subseteq S_1$ for $q' < \overline{q}_1$. Firm 1's indifference condition requires $[q' + \Gamma, \overline{q}_1 + \Gamma] \subseteq S_2$ and $F_2(\overline{q}_2) > F_2(\overline{q}_1 + \Gamma) = r'(\overline{q}_2 - \Gamma)$, and Firm 2's local optimality condition requires $\overline{q}_2 = q* > \overline{q}_1 + \Gamma$. Therefore, Firm 2's indifference condition is contradicted by

$\pi_2(q^*) - \pi_2(\overline{q}_1 + \Gamma) = q^* - \overline{q}_1 - \Gamma - [r(q^*) - r(\overline{q}_1 + \Gamma)] = \max_{q \geq 0}\{q - r(q)\} - [\overline{q}_1 + \Gamma - r(\overline{q}_1 + \Gamma)] > 0.$
■

Lemma A3: Neither S_1 nor S_2 contains an open interval.

Proof: Suppose to the contrary that $[q', q''] \subset S_1$ and $[q'', \overline{q}_1] \cap S_1$ is discrete, for $q' < q''$.

Local optimality requires $F_2(q_2) = r'(q_2 - \Gamma)$ for $q_2 \in [q' + \Gamma, q'' + \Gamma] \subseteq S_2$.

Furthermore, $[q' + \Gamma, q'' + \Gamma] \cap S_2$ also is discrete. Otherwise, $[q''' + \Gamma, q''' + \varepsilon + \Gamma] \subseteq S_2$ for some $q''' > q''$ and $\varepsilon > 0$, and Firm 2's local optimality condition requires $[q''', q''' + \varepsilon] \subseteq S_1$. This contradicts the supposition that $[q'', \overline{q}_1] \cap S_1$ is discrete.

Therefore, S_1 does not contain an open interval.

If, S_1 does not contain an open interval, than neither does S_2 by Local Optimality. ■

Given that S_1 and S_2 are discrete sets, Local Optimality (Proposition 3) implies that the supports have the same cardinality.

Lemma A4: $\# S_1 = \# S_2$.

Proof: If $\# S_1 > \# S_2$, then $F_2(q_1 + \Gamma) = F_2(q_1' + \Gamma)$ for some $q_1, q_1' \in S_1$, $q_1 \neq q_1'$, in which case $r'(q_1) \neq r'(q_1')$ violates Local Optimality. ■

Therefore the supports of equilibrium strategies are $S_1 = \{q_1^1, ... q_1^n\}$ and $S_2 = \{q_2^1, ... q_2^n\}$, for some positive integer *n*. Moreover, without loss of generality, the elements of these sets are strictly monotone, i.e. $q_i^k < q_i^{k+1}$ for $k = 1, ... n+1$.

Local Optimality further implies that in equilibrium consumers are never indifferent between the two products.

Lemma A5: If $q_1 \in S_1$ and $q_2 \in S_2$, then $q_1 + \Gamma \neq q_2$.

Proof: Suppose to the contrary that $q_1 + \Gamma = q_2$ and $F_2(q_1 + \Gamma) = r'(q_1)$. Then $F_2(q_1 + \Gamma - \varepsilon) < r'(q_1 - \varepsilon)$ for $\varepsilon > 0$ sufficiently small, and Firm 1 could increase its profit by investing slightly less. ■

Local optimality further implies that equilibrium S_i do not contain quality choices that have same probability of winning the market. Thus the elements of the two supports necessarily fit together like a zipper.

Lemma A6 ("Zipper Principle"): Either $q_1^k + \Gamma > q_2^k$ for $k = 1, ... n$, or $q_1^k + \Gamma < q_2^k$ for $k = 1, ... n$.

Proof: $q_1^{k+1} + \Gamma > q_2^{l+1} > q_2^l > q_1^k + \Gamma$ or $q_2^{l+1} - \Gamma > q_1^{k+1} > q_1^k > q_2^l - \Gamma$ would violate local optimality by an argument similar to that proving that $\# S_1 = \# S_2$.■

Finally, the Zipper Principle and Local Optimality pin down certain boundary points of the support. There are two kinds of equilibria. If Firm 1 is the "leader", then Firm 1 chooses $q_1 = 1$ with positive probability, and Firm 2 chooses $q_2 = 0$ with positive probability. These roles are reversed if Firm 2 is the leader.

Lemma A7: Either $q_1^n = 1$ and $q_2^1 = 0$, or $q_1^1 = 0$ and $q_2^n = 1$.

Proof: There are two kinds of equilibria by the Zipper Principle. If $q_1^k + \Gamma > q_2^k$ for $k = 1,...n$, then $F_2(q_1^n + \Gamma) = 1$ and Local Optimality requires $q_1^n = 1$; similarly, $F_1(q_2^1 - \Gamma) = 0$ implies $q_2^1 = 0$. A converse argument applies if $q_2^{l+1} - \Gamma > q_1^{k+1} > q_1^k > q_2^l - \Gamma$.■

The above seven lemmas combine to establish the proposition.

---

[i] More specifically, there exists a solution at $\delta = 0$ as shown above, and the solution is differentiable at $\delta = 0$, implying a nearby solution for δ sufficiently small.