COMPETITION FOR A MAJORITY

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Abstract. We define the class of two-player zero-sum games with payoffs having mild discontinuities, which in applications typically stem from how ties are resolved. For games in this class we establish sufficient conditions for existence of a value of the game, maximin and minimax strategies for the players, and a Nash equilibrium. We prove first that if all discontinuities favor one player then a value exists and that player has a maximin strategy. Then we establish that a general property called payoff approachability implies that the value results from an equilibrium. We prove further that this property implies that every modification of the discontinuities yields the same value; in particular, every modification has epsilon-equilibria.

We apply these results to models of elections in which two candidates propose policies and a candidate wins election if a weighted majority of voters prefer his policy. We provide tie-breaking rules and assumptions about voters' preferences sufficient to imply payoff approachability, hence existence of equilibria, and each other tie-breaking rule yields the same value and has epsilon-equilibria. The assumptions are satisfied by generic preferences if the dimension of the space of policies is as large as the number of voters. For games with large electorates, payoff approachability can be verified directly, with no restriction on the dimensionality of the space of policies. These conclusions are then applied to the special case of Colonel Blotto games in which each candidate allocates his available resources among several constituencies and the assumption on voters' preferences is that a candidate gets votes from those constituencies allocated more resources than his opponent offers. Moreover, for simple-majority rule we prove existence of an equilibrium that has zero probability of a tie.

1. Introduction

Following Downs [9], studies of elections often use models in which two candidates compete for votes via the policies they propose. Each candidate's sole objective is to obtain a majority of votes, where each voter will cast her vote for the candidate whose policy she prefers. Because only one candidate can win a majority of votes, such models induce zero-sum games between the candidates. However, because outcomes depend on how voters resolve ties between candidates' policies, the candidates' payoffs are discontinuous functions of their

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policies. A major hindrance to studies of such models has been a lack of sufficient conditions for existence of a value of the game, and existence of maximin/minimax or equilibrium strategies for the candidates.[1] Here we establish such conditions for a large class of games, and then apply them to models in which a candidate must win a weighted or simple majority of votes to win election.

Section 2 defines the class of two-player zero-sum games with payoffs with mild discontinuities, as specified by Assumption 2.1, and establishes two general existence theorems.[2] Section 3 and 4 apply these theorems to voting games in which the winner is determined by majority rule. These games typically have mild discontinuities at strategy profiles where voters indifferent between the policies proposed by the candidates are pivotal in determining the outcome of the election.[3]

Throughout we say that a strategy is *optimal* for player 1 if it is a maximin strategy, or for player 2 if it is a minimax strategy. The general results in Section 2 consider two cases. First we show that if discontinuities invariably favor one player then a value exists and that player has an optimal strategy. This case arises in applications when one candidate wins all ties among voters; or in a legislative context, when the status quo is the default outcome in the event of a tie. Next we identify a general property called payoff approachability.[4] We show that this property implies the condition called 'better-reply security' by Reny [23], which then implies that the players have equilibrium strategies that yield the value. Moreover, we show that in games satisfying payoff approachability this remains the value for every modification of payoffs at discontinuities. That is, if there exists some tie-breaking rule for which the payoff function satisfies payoff approachability then in fact the value is invariant to tie-breaking rules.

In the applications to models of elections, therefore, we show that the value exists and is independent of tie-breaking rules by verifying that payoff approachability is satisfied by

[1]In a two-player zero-sum game, maximin and minimax payoffs are defined in terms of supremum and infimum operators applied to a player's payoffs, and when these two payoffs are the same the game is said to have a (unique) value. If equilibrium strategies exist then the value is player 1's equilibrium payoff. A maximin strategy for player 1 is a strategy that assures him at least the value for every strategy of player 2, and a minimax strategy for player 2 is one that holds player 1's payoff down to the value. More generally, whenever the value exists each player has an ε-optimal strategy for every $\varepsilon > 0$, and a profile of these strategies is an ε-equilibrium.

[2]Other than Dasgupta and Maskin [7] and Parthasarathy [21], who focus on discontinuities along well-behaved curves with zero measure, the prior literature has not restricted the set of strategies where payoffs are discontinuous and therefore must allow for pervasive discontinuities.

[3]Although other games of economic interest, such as auctions and Bertrand-style competition between duopolists, have payoffs with mild discontinuities, we do not address them here because typically the payoffs are not zero-sum.

[4]The terminology is suggestive of what the condition requires. It is not to be confused with Blackwell's [3] approachability of a set of players' payoffs in a repeated game described in footnote 14.

a particular rule that is convenient for the verification. In several cases this is not the 'standard' tie-breaking rule that resolves each tie by tossing a fair coin. Nevertheless, this method suffices to obtain the general result — the value exists and is the same for every tie-breaking rule. Moreover, existence of the value implies that for every $\varepsilon > 0$ each candidate has a strategy that assures a payoff within ε of the value, and thus an ε-equilibrium exists.

Section 3 applies these results to models of elections. Although a contest between two candidates might proceed dynamically, we consider only the normal form induced by the game in extensive form. Candidates compete for election by offering policies. If no voter is indifferent between the candidates' offered policies then each voter casts her vote for the candidate whose policy she prefers, and the candidate elected is the one winning a majority of votes.[5] Using the results established in Section 2, we show that if one candidate wins all ties then the value exists and that candidate has an optimal strategy. For another tie-breaking rule that is symmetric, we identify assumptions on voters' preferences sufficient to imply payoff approachability and thus better-reply security, ensuring that the candidates have equilibrium strategies that yield the value, and any other tie-breaking rule yields the same value and has epsilon-equilibria. If the dimension of the space of policies is as large as the number of voters then these assumptions are satisfied by voters' preferences that are generic within a large class. In the case of a continuum of voters, we identify a simple tie-breaking rule under which the assumptions on voters' preferences are also satisfied generically, regardless of the dimension of the policy space. It follows that games with sufficiently large electorates have epsilon-equilibria for any tie-breaking rule and that such games can be well approximated by games with a continuum of voters that have equilibria.

Section 4 obtains stronger results for the special case of 'Colonel Blotto' games with weighted-majority rule, which are often used to model election campaigns and lobbying.[6] In these games a candidate's policy allocates his available resources among several constituencies, each of which votes for the candidate offering more. As in Section 3, such a game has a value when one of the candidates wins all ties, and this candidate has an optimal strategy. To address other cases we provide a tie-breaking rule that implies payoff approachability. Applying our general results to games with this tie-breaking rule shows that the candidates

[5] See Banks, Duggan, and Le Breton [2] for a synopsis of the prior literature, and their proof under weak assumptions that if a two-player symmetric zero-sum game has an equilibrium then its support lies within a subset of policies called the uncovered set. Banks and Duggan [1] study a dynamic model with reputation effects in which announced policies are 'cheap talk' that can differ from candidates' actual preferences and implemented policies.

[6] The moniker *Colonel Blotto* stems from the paper by Gross and Wagner [13], but studies of such games date to work in 1921 by Borel; cf. Borel [5] and Borel and Ville [6]. Most analyses of such games assume that each player's objective is to maximize the number of votes won, as in Roberson [24], rather than winning a majority of votes as assumed here.

have equilibrium strategies that yield the value, and games with any other tie-breaking rule inherit this value. For the special case that the winner is the candidate obtaining a simple majority of votes, we show that the value results from an equilibrium that has zero probability of ties.

1.1. **Synopsis and Relation to the Literature.** The sequel has two parts: Section 2 defines the class of zero-sum games with mild discontinuities and obtains two new general existence theorems. Theorem 2.6 shows that a game has a value if all discontinuities are resolved in favor of one player, and that player has a maximin strategy. Theorem 2.9 shows that payoff approachability implies existence of an equilibrium, and the value is invariant across all other resolutions of discontinuities. These theorems are applied in Sections 3 and 4 to models of elections in which the candidate who wins a weighted-majority of votes wins election. In these applications, mild discontinuities occur where a voter is indifferent between the two candidates' proposed policies.

Section 3 considers an election in which two candidates propose policies in an N-dimensional space over which the K voters have preferences, and the winner is determined by weighted-majority rule. Theorem 2.6 implies that if all ties are resolved in favor of one candidate then the value and ε-equilibria exist. This result is new in the literature on voting games, and tie-breaking in favor of an incumbent or status quo is observed in practice. To establish existence of an equilibrium, we provide an explicit tie-breaking rule and conditions on voters' preferences sufficient to imply payoff approachability. Moreover, the value is invariant to tie-breaking rules, so ε-equilibria always exist, and we show further that equilibria and the value are limits of those obtained by approximating finite games.[7] If $N \geqslant K - 1$ and the policy space is the convex hull of the voters' ideal points, then these conditions are satisfied generically by voters' preferences represented by concave differentiable utility functions. Although restrictive, these conditions improve on the few results in the literature.[8] Moreover, as we provide examples of simple games with $N = 2$ and $K = 3$ where a value does not exist, there is not much room for improving on the conditions that we identify. Nevertheless, on the other extreme of a large number of voters, payoff approachability is verified using

[7]Invariance of the value resembles the result of Jackson and Swinkels [15] for auctions with an atomless distribution of bidders' private values, for which they show existence of equilibria invariant to the tie-breaking rule; in particular, because some equilibria are invariant, to prove existence one can rely on endogenous sharing as in Simon and Zame [26]. For the electoral model studied here, we obtain the weaker result that, when payoff approachability is satisfied, it is the value that is invariant, not necessarily the strategies. This invariance of the value follows from payoff approachability, whereas in an auction, existence of invariant strategies stems from the assumed atomless distribution of bidders' valuations.

[8]Duggan [10] shows existence for the case of a convex policy space and $N = 2$ and $K = 3$; and Duggan and Jackson [12] allow more general conditions on voters' preferences but rely on endogenous tie-breaking to show existence of equilibria for simple-majority rule.

the symmetric tie-breaking rule of zero payoff in case of ties, without any restriction on the dimension N.

Section 4 considers the special case of Colonel Blotto games with majority rule. The results of Section 3 apply fully to this case since $N = K - 1$ and voters' utilities are linear. Thus we obtain a complete solution to the problem of existence of equilibria and a value. The only prior result for majority rule is by Duggan [11], who assumes simple-majority rule and symmetric resources, whereas we allow weighted-majority rule and asymmetric resources. We show further that the value is invariant to the tie-breaking rule, so ε-equilibria always exist, and for simple-majority rule, we show existence of an equilibrium for which the probability of ties is zero. The other literature on Colonel Blotto games assumes plurality rule or other simpler rules to which our results also apply.

2. General Existence Theorems

We start by presenting general results for zero-sum games. Appendix A provides the proofs omitted in this section.

The reader interested in particular results for spatial models of elections and Colonel Blotto models can either skip to Sections 3 and 4, respectively, or appreciate the results that follow without studying their details.

We study two-player zero-sum games with the following general features. In each game, the two players are labeled by $i = 1$ and 2. Given a player i, let j be the other player. For each player i, his set X_i of pure strategies is a compact metric space and his set Σ_i of mixed strategies is the set of Borel probability measures on X_i endowed with the weak-$*$ topology.[9] Since X_i is a compact metric space, so is Σ_i.

Let $X = X_1 \times X_2$ and $\Sigma = \Sigma_1 \times \Sigma_2$ denote the sets of profiles of players' pure and mixed strategies. Player i's payoff function from pure strategies is a Borel-measurable function $\pi_i : X \to [-1, +1]$, where $\pi_1 + \pi_2 = 0$, and it is extended to the expected payoff from mixed strategies via $\pi_i(\sigma_1, \sigma_2) = E_{\sigma_1,\sigma_2}[\pi_i(x_1, x_2)]$ for each profile $(\sigma_1, \sigma_2) \in \Sigma$.[10] Recall that when $(\sigma_1^n, \sigma_2^n) \to (\sigma_1, \sigma_2)$, the corresponding product measure $\sigma_1^n \otimes \sigma_2^n$ converges to $\sigma_1 \otimes \sigma_2$. So $E_{\sigma_1,\sigma_2}[f(x_1, x_2)]$ is upper semi-continuous (u.s.c.) if $f : X \to \mathbb{R}$ is u.s.c., and l.s.c. if f is l.s.c.

Let $D \subset X$ be the subset consisting of those pure-strategy profiles at which π_1 and π_2 are discontinuous. D is a Borel measurable set (cf. Billingsley [4, Appendix M10]). We focus on

[9] It is sufficient that the strategy sets be compact. Metrizability is assumed to simplify the exposition of the proofs by allowing us to use sequences rather than nets. See Remark A.3(2) for more on this.

[10] The restriction to $[-1, +1]$ as the range is without loss if the payoff functions are bounded. But all that we require is that the payoffs are bounded in a neighborhood of the set of discontinuities; see also Remark 2.3.

games for which D is not empty, although we do not assume it explicitly. For each player i and his pure strategy $x_i \in X_i$, let $D(x_i) \subset X_j$ be the cross-section of D at x_i, i.e. the set of x_j such that $(x_i, x_j) \in D$. The class of games with *mild discontinuities* consists of those that satisfy the following assumption.

Assumption 2.1 (Mild Discontinuities)**.** For each player i the set $\{ \sigma_i \in \Sigma_i \mid \sigma_i(D(x_j)) = 0 \ \forall x_j \in X_j \}$ is dense in Σ_i.

For future reference, we record one implication of Assumption 2.1:

Lemma 2.2. *Assumption 2.1 implies that for each mixed strategy σ_j of a player j the set $\{ x_i \in X_i \mid \sigma_j(D(x_i)) = 0 \}$ is dense in X_i.*

Remark 2.3. Standard existence theorems for zero-sum games—cf. Mertens [20] and Reny [23]—require only that the strategy sets be compact, and that the payoff function (in mixed strategies) of each player be upper semi-continuous in his strategy. If we define D to be the set of profiles where upper semi-continuity fails in the strategy of at least one of the players, then our results go through, even allowing for the payoffs to be unbounded, as long as the payoffs are bounded in a neighborhood of D.

Say that a pure strategy $x_i \in X_i$ of player i is a *point of continuity* against the other player j's mixed strategy $\sigma_j \in \Sigma_j$ if σ_j assigns zero probability to the cross section $D(x_i)$. At such a pair of strategies, player i's expected payoff $\pi_i(x_i, \sigma_j)$ is independent of how payoffs are determined at profiles in $D(x_i)$. The phrase "point of continuity" is justified by Lemma 2.5 below.

The following sufficient condition for Assumption 2.1, which is satisfied in many typical games, is readily verifiable.

Lemma 2.4. *If X_j is a finite dimensional manifold then Assumption 2.1 holds if, for each $x_i \in X_i$, $D(x_i)$ is a set of lower dimensionality in X_j.*[11]

We consider a basic game and the corresponding family of games that differ only in their payoffs at profiles in D, which in applications correspond to the possible resolutions of ties. Represent the basic game as $G(\pi)$ where $\pi_1 = \pi$ and $\pi_2 = -\pi$. Variants of this basic game are parameterized by the set Π of payoff functions $\pi' : X \to [-1, 1]$ such that $\pi'(x) = \pi(x)$ for all $x \notin D$. Thus the family of games is $\{G(\pi') \mid \pi' \in \Pi\}$.

[11] If X_j is in addition a subset of an Euclidean space, then Assumption 2.1 is satisfied if the Lebesgue measure of $D(x_i)$ is zero for each $x_i \in X_i$.

Let δ_{x_i} denote the Dirac measure in Σ_i concentrated at $x_i \in X_i$. Observe that $\pi_i(x_i,\cdot) = \pi_i(\delta_{x_i},\cdot)$, so we shall use both notations for expositional purposes. The following is a direct application of the Mapping Theorem (Billingsley [4, Theorem 2.7]).

Lemma 2.5. *If $\sigma_j(D(x_i)) = 0$ then player i's payoff function π'_i is continuous at $(\delta_{x_i}, \sigma_j) \in \Sigma_i \times \Sigma_j$.*

For each payoff function $\pi' \in \Pi$, player 1's *maximin* and *minimax* values are

$$\underline{v}(\pi') = \sup_{\sigma_1 \in \Sigma_1} \inf_{x_2 \in X_2} \pi'(\sigma_1, x_2) \qquad \text{and} \qquad \overline{v}(\pi') = \inf_{\sigma_2 \in \Sigma_2} \sup_{x_1 \in X_1} \pi'(x_1, \sigma_2)\,,$$

where necessarily $\underline{v}(\pi') \leqslant \overline{v}(\pi')$. If $\underline{v}(\pi') = \overline{v}(\pi') \equiv v^*(\pi')$ then $v^*(\pi')$ is called the *value* of the game $G(\pi')$ to player 1.

2.1. **The Case that One Player Wins All Ties.**

Of particular interest are the two games $G(\pi^+)$ and $G(\pi^-)$ defined as follows: $\pi_1^+(x) = +1$ and $\pi_1^-(x) = -1$, for each profile $x \in D$. In applications these correspond to the two cases where one player wins all ties, or in a legislative context that the status quo is the default outcome in the event of a tie. The following theorem establishes existence of values for these games.

Theorem 2.6. *If $\pi' = \pi^+$ or $\pi' = \pi^-$ then the value $v^*(\pi')$ exists. Moreover, in the game $G(\pi^+)$ player 1 has a maximin strategy, and in the game $G(\pi^-)$ player 2 has a minimax strategy.*

The proof uses the fact that mild discontinuities allows one to approximate the game with continuous games that have values and optimal strategies. Because the approximating strategy of player 2 is feasible in the given game, the limiting value v^* cannot be smaller than the minimax value $\overline{v}(\pi^+)$. By mild discontinuities again, and by the asymmetry of the tie-breaking rule, player 1 can guarantee v^* using the limiting strategy σ_1^*.

Remark 2.7. Define the u.s.c. and l.s.c. payoff functions $\overline{\pi}_i, \underline{\pi}_i : X \to \mathbb{R}$ by $\overline{\pi}_i(x) \equiv \sup_{x^n \to x} \limsup_n \pi_i(x^n)$ and $\underline{\pi}_i(x) \equiv \inf_{x^n \to x} \liminf_n \pi_i(x^n)$. Then $\overline{\pi}_i \geqslant \pi_i \geqslant \underline{\pi}_i$ with equalities on $X \setminus D$. Let $\overline{\Pi}$ and $\underline{\Pi}$ be the sets of u.s.c. and l.s.c. functions in Π that majorize $\overline{\pi}$ and minorize $\underline{\pi}$, respectively. The optimal strategy of player 1 in π^+ derived in the proof of Theorem 2.6 is an optimal strategy in each game in $\overline{\Pi}$, and the value $v^*(\pi^+)$ is the value of each game in $\overline{\Pi}$. Hence the tie-breaking rule favoring player 1 could be the rule that generates a game $\overline{\pi} \in \overline{\Pi}$ — and analogously for each game $\underline{\pi} \in \underline{\Pi}$.

2.2. **A Sufficient Condition for Existence of an Equilibrium.** A game that has a value has an ε-equilibrium σ^ε for every $\varepsilon > 0$. Also, if it has a value and player i has an optimal strategy σ_i^*,[12] then for every $\varepsilon > 0$, σ_i^ε can be chosen to be σ_i^*. While Theorem 2.6 shows that two specific variants of a game have a value, ideally one wants an existence result that does not depend on how ties are resolved. To obtain such an invariance result, we invoke the following sufficient condition.[13]

Definition 2.8 (Payoff Approachability)**.** A payoff function $\tilde{\pi} \in \Pi$ satisfies *payoff approachability* if for each player i, his pure strategy $x_i \in X_i$, and the other's mixed strategy $\sigma_j \in \Sigma_j$,

$$\tilde{\pi}_i(x_i, \sigma_j) \leqslant \sup_{x_i^n \to x_i} \limsup_n \tilde{\pi}_i(x_i^n, \sigma_j) ,$$

where the supremum is over all sequences $\{x_i^n\} \subset X_i$ converging to x_i for which each pure strategy x_i^n in the sequence is a point of continuity against σ_j.

Payoff approachability requires that a player's payoff cannot be more than the limit of what he can get from nearby points of continuity against any strategy of his opponent.[14] Roughly, payoff approachability assures that a player can avoid unfavorable ties by moving slightly away from them. In the applications to voting games we specify tie-breaking rules and assumptions on voters' preferences sufficient to imply payoff approachability.

Theorem 2.9. *If there exists a payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability then:*

(1) *$G(\tilde{\pi})$ has an equilibrium that yields the value $v^*(\tilde{\pi})$.*
(2) *For each $\varepsilon > 0$, each player j has a strategy σ_j^ε that is ε-optimal in $G(\tilde{\pi})$ and such that $\sigma_j^\varepsilon(D(x_i)) = 0$ for all $x_i \in X_i$.*
(3) *For each payoff function $\pi' \in \Pi$, the value $v^*(\pi')$ exists and is the same as $v^*(\tilde{\pi})$; that is, the value is invariant.*

As in the proof of Theorem 2.6, we use well behaved approximating games to construct candidates v^* and σ^*, and obtain $\tilde{\pi}(x_1, \sigma_2^*) \leqslant v^*$ for points of continuity against σ_2^*. Then payoff approachability allows extends the inequality to all $x_1 \in X_1$ because the discontinuity cannot favor player 1 so much as to reverse this inequality. Moreover, by construction,

[12]That is, a maximin strategy if $i = 1$ and a minimax strategy if $i = 2$.

[13]Observe that for any payoff function π', $v^*(\pi^-) = \underline{v}(\pi^-) \leqslant \underline{v}(\pi') \leqslant \overline{v}(\pi') \leqslant \overline{v}(\pi^+) = v^*(\pi^+)$, so the value is independent of π' iff $v^*(\pi^-) = v^*(\pi^+)$. Payoff approachability ensures this last equality.

[14]We use the name payoff approachability to distinguish it from Blackwell's [3] definition for repeated games of approachability of a subset of the players' pairs of possible long-run average payoffs, which requires that for some mixed strategy of one player and any mixed strategy of the other player, eventually the resulting sequence of time-average payoffs is arbitrarily close to the set with arbitrarily high probability. The restriction to nearby points that are points of continuity against σ_j implies that we could have used π_i instead of $\tilde{\pi}_i$ in the right-hand side of the above inequality.

equilibria of approximating games have no ties, so player 2 can hold player 1 down to $v^*(\tilde{\pi})$ for any tie-breaking rule.

Remark 2.10.

(1) Although part (3) establishes that if some payoff function $\tilde{\pi} \in \Pi$ satisfies payoff approachability then for every payoff function $\pi' \in \Pi$ the game $G(\pi')$ has a value $v(\pi') = v(\tilde{\pi})$, it need not be in $G(\pi')$ that a player has an optimal strategy, or if he does then it could depend on the tie-breaking rule; see Remark 4.5 below for an example. Nevertheless, the proof establishes that for each $\varepsilon > 0$ player 1 has a strategy that assures at least $v^*(\pi') - \varepsilon$ regardless of the tie-breaking rule. Even if no payoff function in Π satisfies payoff approachability, it is still possible that for every payoff function $\pi' \in \Pi$ the game $G(\pi')$ has an equilibrium and a value, but the value depends on the tie-breaking rule. An example is the "diagonal game" at the end of Section 2.3 below.

(2) In some applications, some strategies may be (weakly) dominated and payoff approachability seems irrelevant for these profiles. The hypothesis of Theorem 2.9 can be weakened as follows. Suppose each player i has a compact subset X_i^* of X_i such that for each $x_i \in X_i$, there exists $x_i \in X_i^*$ such that $\tilde{\pi}_i(x_i^*, \sigma_j) \geqslant \tilde{\pi}_i(x_i, \sigma_j)$ for all σ_j. Then for the conclusion of Theorem 2.9 to hold it is sufficient that payoff approachability holds for all $x_i \in X_i^*$ for each player i.

(3) If payoff approachability holds just for just one player i, in the sense that it holds for all (x_i, σ_j) for fixed i and j, then the game has a value and player j has an optimal strategy. For instance, this happens in the games π^+ for $i = 2$ and π^- for $i = 1$.

Lemma A.2, from the proof of Theorem 2.9, yields the following corollary about finite approximations. Recall that every two-player zero-sum game with finite sets of pure strategies has a value obtained from equilibrium strategies that can be computed by linear programming.

Corollary 2.11. *Suppose there exists a payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability. Consider a sequence of finite games $G(\tilde{\pi}^n)$, where $\tilde{\pi}^n$ is the restriction of $\tilde{\pi}$ to profiles in $\Sigma_1^n \times \Sigma_2^n \subset \Sigma$, with Σ_i^n being the set of mixed strategies over the finite set of strategies $X_i^n \subset X_i$, and each sequence Σ_i^n converges to Σ_i in the Hausdorff topology on closed subsets. Let σ^n and v^n be an equilibrium and the value of $G(\tilde{\pi}^n)$ for each n. Then every limit point of σ^n is an equilibrium of $G(\tilde{\pi})$, and v^n converges to $v^*(\tilde{\pi})$.*

This corollary has practical implications for computation. It implies that the linear program for a finite approximation of a game satisfying payoff approachability for some tie-breaking rule one can use pure-strategy profiles in $X \setminus D$ at which payoffs are continuous, independently of the actual tie-breaking rule.

We conclude this subsection with a sufficient condition for a payoff function $\tilde{\pi}$ to satisfy payoff approachability. The simplification achieved by this result is that in a class of games, which includes our subsequent applications, it is enough to check whether payoff approachability holds against mixed strategies with finite support. More precisely, if a payoff function satisfies condition (1) of Proposition 2.12 below, then payoff approachability is equivalent to condition (2).

Proposition 2.12. *A payoff function $\tilde{\pi} \in \Pi$ satisfies payoff approachability if:*

(1) *For each i, x_i, $D(x_i)$ can be partitioned into finitely many Borel-measurable subsets $D^1(x_i), \ldots, D^n(x_i)$ such that for each $1 \leqslant l \leqslant n$:*
   (a) *$\tilde{\pi}_i(x_i, \cdot)$ is constant on $D^l(x_i)$.*
   (b) *For each closed $A^l \subseteq D^l(x_i)$, $\tilde{\pi}_i(y_i, \cdot)$ is constant on A^l for an open and dense set of y_i's in a neighborhood U of x_i.*

(2) *The condition in Definition 2.8 of payoff approachability holds for i, x_i and σ_j where the support of σ_j is finite and contained in $D(x_i)$.*

2.3. **Relation to Reny's Conditions.** Theorem 2.6 adds to the literature on sufficient conditions for existence of equilibria. To see this consider the following example.

Example 2.13. The sets of pure strategies are $X_1 = X_2 = [0, 1]$ and player 1's payoff is

$$\pi_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 < x_2 \\ 1 - x_1 & \text{if } x_1 > x_2 \\ 1 & \text{if } x_1 = x_2 \end{cases}$$

The set D is the diagonal $x_1 = x_2$, and this is the π^+ version, where player 1 gets $+1$ on D. Consider a sequence of profiles $(\delta_{1/2}, \delta_{1/2+1/n})$. The profile of payoffs along the sequence is $(1/2, -1/2)$. The sequence converges to the profile $(\delta_{1/2}, \delta_{1/2})$ with associated profile of payoff limits $(1/2, -1/2)$. The limiting profile is not an equilibrium, as player 2 gets -1 and could get $-1/2$ by any $x_2 \neq 1/2$. Better-reply security fails: if σ_1 is a strategy of player 1, for each ε, we can choose a point $x_2(\varepsilon)$ in the interval $(1/2 - \varepsilon, 1/2)$ that is a point of continuity against σ_1 and $\pi_1(\sigma_1, x_2(\varepsilon)) \leqslant 1/2 + \varepsilon$. Likewise, against $\delta_{1/2}$, player 2 gets -1 for $x_2 = 1/2$ and $-1/2$ for any $x_2 \neq 1/2$, so $\pi_2(\delta_{1/2}, \sigma_2) \leqslant -1/2$ for all $\sigma_2 \in \Sigma_2$. Thus no strategy of either player can secure strictly more than the corresponding payoff limit. Yet Theorem 2.6

establishes existence of a value and of a maximin strategy for player 1. (It is directly verified that the value of the game is 1/2 and (σ_1, σ_2), with $\sigma_1 = \delta_{1/2}$ and $\sigma_2 = (1/2)\delta_0 + (1/2)\delta_1$, is an equilibrium.) See Section 4 for another example.

On the other hand, the direction taken by Theorem 2.9 is evidently a specialization of better-reply security.[15] To illustrate, we note that its proof fails in the standard example of a zero-sum game without a value due to Sion and Wolfe [27] because this game violates payoff approachability.

Example 2.14 (Sion-Wolfe Example of a Game with No Value)**.** There are two players, with strategy sets $X_1 = X_2 = [0, 1]$. Player 1's payoff is

$$\pi_1(x_1, x_2) = \begin{cases} -1 & \text{if } x_1 < x_2 < x_1 + 1/2 \\ 0 & \text{if } x_2 = x_1 \text{ or } x_2 = x_1 + 1/2 \\ 1 & \text{otherwise} \end{cases}$$

If we take $x_1 = 0$ and $\sigma_2 = \delta_{1/2}$ then $\pi_1(x_1, \sigma_2) = 0$, while $\pi_1(x_1^n, \sigma_2) = -1$ when we take a sequence of points of continuity. A similar situation holds for $x_1 = 1$ and $\sigma_2 = \delta_1$. The fundamental problem is that these are boundary points for player 1 and one can approach such a point from only one side. By the same logic, there is no payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability. In fact, payoff approachability forces $\tilde{\pi}_1(1, 1) = -1$, as $\pi_1(x_1^n, 1) = -1$ for all sequences $x_1^n \to 1$, and also $-\tilde{\pi}_1(1, 1) = \tilde{\pi}_2(1, 1) = -1$, as $\pi_2(1, x_2^n) = -1$ for all sequences $x_2^n \to 1$.

The same logic applies even to the better-reply secure "diagonal game" for which π_1 is -1 if $x_2 > x_1$, $+1$ if $x_1 > x_2$, and 0 if $x_1 = x_2$. More generally, such a game has a pure-strategy equilibrium $(x_1, x_2) = (1, 1)$ yielding the value $v \in [-1, +1]$ when a tie-breaking rule specifies $\pi_1 = v$ on the diagonal where $x_1 = x_2$. Because the value v depends on the tie-breaking rule that specifies v, there cannot exist a payoff function $\tilde{\pi} \in \Pi$ that satisfies payoff approachability.

We note that for a general n-player, non zero-sum game $(X_i, \pi_i)_{i=1}^n$, payoff approachability and mild discontinuities, together with $\sum_{i=1}^n \pi_i$ upper semicontinuous in $X = \times_{i=1}^n X_i$, provide a straightforward extension of the conditions in Dasgupta and Maskin [7] for general strategy spaces.[16] Of course, Reny's payoff security and reciprocal upper semicontinuity subsume all such conditions.

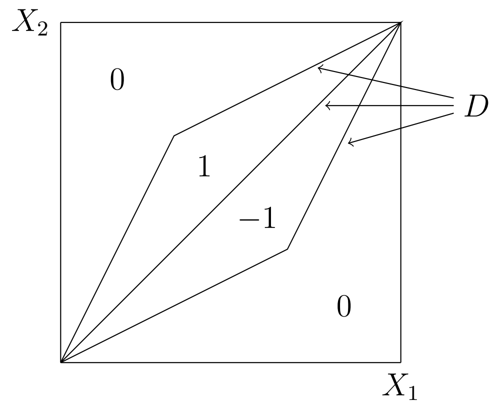
[15] Indeed, payoff approachability is a specialization of payoff security. We note that Reny's theorem is sometimes difficult to apply, especially to show existence of a mixed-strategy equilibrium. So the specialization to payoff security is not an issue.

[16] Dasgupta and Maskin [7] only allowed for one-dimensional strategy spaces.

Finally, let us illustrate invariance of the value when there is one $\tilde{\pi} \in \Pi$ satisfying payoff approachability.

Example 2.15. Again, consider a two-player game with $X_1 = X_2 = [0,1]$, and player 1's payoff described in Figure 1 below:

Figure 1



Setting $\tilde{\pi}_1 = 0$ at D, payoff approachability is readily verified and $x_1 = x_2 = 0$ is a Nash equilibrium, yielding the value $v(\tilde{\pi}) = 0$. Now consider π^+, with $\pi_1^+ = 1$ at D. Better reply security is violated at $(0,0)$, as it is not a Nash equilibrium and no player can secure strictly more than the payoff limit of zero. Yet, as $\tilde{\pi} \in \Pi$ satisfies payoff approachability, the value of π^+ (which must exist from Theorem 2.6) must be equal to zero from Theorem 2.9. In fact, $\underline{v}(\pi^+) \geqslant 0$, as $x_1 = 0$ yields at least zero; for each integer k, let σ_2^k be the simple average of $k+1$ uniform distributions over the intervals $[0, 2^{-k}/3], [0, 2^{-(k-1)}/3], ..., [0, 1/3]$. Then one verifies that $\pi_1(x_1, \sigma_2^k)$ is at most of order $1/k$ for every $x_1 \in [0,1]$, establishing that $\overline{v}(\pi^+) \leqslant 0$ and thus $v(\pi^+) = 0$. It also establishes that player 1 has a maximin strategy, whereas player 2 need not have one (i.e. $(x_1 = 0, \sigma_2 = \sigma_2^k)$ is an $1/k$-equilibrium for every k.) The symmetric argument verifies that $v(\pi^-) = 0$, so the value exists for every $\pi' \in \Pi$ and is equal to zero.

3. Models of Elections

In this section we address models of elections, as in Downs [9]. Each candidate proposes a policy and gets votes from those voters who prefer his policy to the policy proposed by the other candidate. First we apply Theorem 2.6 to conclude that if one candidate, say the incumbent, wins all ties then a value exists and the incumbent has an optimal strategy that ensures this value. Then, invoking assumptions on voters' preferences, we show that payoff approachability is satisfied for a specified tie-breaking rule. Therefore, Theorem 2.9 implies existence of an equilibrium that yields the value, and this is also the value for any other tie-breaking rule (so there exists an ε-equilibrium for every $\varepsilon > 0$).[17]

3.1. **Formulation.** The game G is specified as follows. Two candidates compete in an election for the votes of K voters, where $K > 2$, by choosing a policy from a set P of feasible policies. Specifically, each candidate i's set of feasible policies is a subset $X_i \subset P$, and the set of feasible profiles of policy proposals is $X = X_1 \times X_2$.[18] If voter k chooses candidate i then i gets w_k votes, where each $0 < w_k < 1/2$ and $\sum_k w_k = 1$. A candidate who gets more than half the votes wins election and receives the payoff $+1$, and the loser receives the payoff -1. In the case of a draw, in which the candidates get equal numbers of votes, their payoffs are both zero. As in Section 2, the payoff function of candidate i is $\pi_i : X \to [-1, +1]$, which can depend on how voters choose between tied policies.

Each voter is assumed to choose the candidate whose policy she prefers; that is, only the policies matter, not the identities of the candidates who propose them. We represent voter k's preferences by a utility function $u_k : P \to \mathbb{R}$. Then at a profile $(x_1, x_2) \notin D$ with no ties, $\pi_i(x_1, x_2) = \text{sign}[\sum_{k \in W_i(x_1,x_2)} w_k - 1/2]$ where $W_i(x_1, x_2) = \{k \mid u_k(x_i) > u_k(x_j)\}$ is the set of voters who prefer the policy of candidate i.

We impose the following assumptions on the policy space and the voters' preferences.

Assumption 3.1 (Basic Assumptions)**.**

(1) For each candidate i, his set $X_i \subseteq P$ is a manifold
(2) Each voter's utility function is continuous, and the intersection of each indifference curve with X_i is a lower-dimensional set for each i.

Lemma 3.2. *The game G is mildly discontinuous.*

Proof. For each policy x_i, the cross-section $D(x_i)$ of D is contained in the intersection of X_j with a finite union of indifference sets in P, one for each voter, each of which is a lower-dimensional set, so Assumption 2.1 is satisfied using Lemma 2.4. □

3.2. **The Case That One Candidate Wins All Ties.** Theorem 2.6 implies that if one candidate wins all ties then a value exists and that candidate has an optimal strategy that ensures at least the value.

[17] Plott [22] shows that an equilibrium in *pure* strategies exists only if voters have highly non-generic utility functions. Duggan [10] shows that an equilibrium exists in the case of three voters and the standard tie-breaking rule. Duggan and Jackson [12] show that an equilibrium exists under more general assumptions, but they rely on endogenous tie-breaking as in Simon and Zame [26].

[18] Typically, symmetry is imposed in such models by assuming that $X_1 = X_2$, but our results do not require this assumption. We apply this more general formulation to asymmetric Colonel Blotto games in Section 4.

3.3. **Assumptions on Voters' Preferences.** Now we impose further assumptions on voters' preferences and their strategy sets that, together with the tie-breaking rule specified in subsection 3.5, imply payoff approachability and thus existence of an equilibrium.

Our first assumption assures a unique winner when there are no ties. Remark 3.13(2) shows why this assumption can matter for a weighted-majority game; however, this assumption is omitted in the case of simple-majority games addressed later.

Assumption 3.3. For each subset L of voters, $\sum_{k \in L} w_k \neq 1/2$.

Say that a subset of voters L is a minimal subset of voters for whom x_i is Pareto optimal if: (i) x_i is Pareto optimal for voters in L among the policies in X_i; and (ii) there does not exist a strict subset of L for whom it is Pareto optimal. For each policy $x_i \in X_i$, let $\overline{K}(x_i)$ be the subset of voters for whom x_i is an *ideal* policy in X_i, i.e. x_i maximizes u_k over X_i. Obviously each voter in $\overline{K}(x_i)$ is a singleton minimal set for whom x_i is optimal among policies in X_i. Let K denote both the number and the set of voters.

Assumption 3.4 (Diversity of Preferences)**.** For each candidate i and policy $x_i \in X_i$:

(1) The policy x_i is Pareto optimal in X_i.
(2) For each minimal subset L of voters for whom the policy x_i is Pareto optimal, each voter $k \in L$, and each neighborhood V of x_i, there exists a policy y_i in V such that $u_{k'}(x_i) < u_{k'}(y_i)$ for every voter k' in $K \setminus \overline{K}(x_i)$ other than voter k, while $u_{k'}(x_i) > u_{k'}(y_i)$ for all voters $k' \in \overline{K}(x_i) \cup \{k\}$.

Remark 3.13(1) below shows how to relax Assumption 3.4(1). Assumption 3.4(2) is crucial to establishing payoff approachability, as it allows each candidate to move away from a particular policy and capture all other voters but one, except those voters for which the policy is already ideal. We discuss in subsection 3.4 the restrictions it imposes after introducing one further assumption, but first some additional notation is required.

Observe that there exists at most one minimal subset L of $K \setminus \overline{K}(x_i)$ for whom x_i is Pareto optimal if the assumption is satisfied. Indeed, if there were two such L's, say L and L', then picking $k \in L$, it would be possible to find y_i that is better than x_i for all voters in L', contradicting the Pareto optimality of x_i. Moreover there exists one iff $\overline{K}(x_i)$ is empty. Thus define $K^*(x_i)$ to be $\overline{K}(x_i)$ if the latter is nonempty and otherwise the unique minimal subset L of K for which x_i is Pareto optimal.[19]

[19]If we had assumed that the ideal policies of voters are all different, then each x_i has a unique minimal subset of voters for whom x_i is Pareto optimal. We do not impose this assumption because models with linear preferences over a convex set could admit the robust possibility that the same policy is ideal for multiple voters.

Given a policy $x_i \in X_i$, for every neighborhood $V(x_i)$ of x_i, and every $k \in K^*(x_i)$, from Assumption 3.4 we can choose a policy $y_i(V(x_i), k) \in V(x_i)$ such that $u_{k'}(x_i) < u_{k'}(y_i(V(x_i), k))$ if $k' \neq k$ and belongs to $K \setminus \overline{K}(x_i)$; and $u_{k'}(x_i) > u_{k'}(y_i(V(x_i), k))$ otherwise. To simplify notation, we use y_i^k to denote $y_i(V(x_i), k)$.

If $x = (x_1, x_2) \in D$ then for each i, define $L^i(x)$ as the set of voters k such that $u_k(x_i) > u_k(x_j)$, and $L^0(x)$ as the set of voters k such that $u_k(x_i) = u_k(x_j)$; $L_i^*(x) \equiv K^*(x_i) \cap L^0(x)$; and $\overline{L}_i(x) = \overline{K}(x_i) \cap L^0(x)$. For all sufficiently small neighborhoods $V(x_i)$ of x_i, for each $y_i \in V(x_i)$, $u_k(y_i) > u_k(x_j)$ for all $k \in L^i(x)$ and $u_k(y_i) < u_k(x_j)$ for all $k \in L^j(x)$. Observe that by construction the payoffs are then well-defined without ties for (y_i^k, x_j) for each $k \in K^*(x_i)$. Thus, by Assumption 3.3, $\pi_i(y_i^k, x_j)$ is ± 1.

With this notation, we can introduce our next assumption. Suppose $x \in D$ and that either $\overline{L}_i(x)$ is nonempty or $|L_i^*(x)| \geqslant 2$. Let $l_i^*(x)$ be a voter in $L_i^*(x)$ with $w_{l_i^*(x)} \leqslant w_{k'}$ for all $k' \in L_i^*(x)$. For each i, let $V(x_i)$ be a neighborhood of x_i such that for all $y_i \in V(x_i)$, $u_k(y_i) > u_k(x_j)$ for all $k \in L^i(x)$ and $u_k(y_i) < u_k(x_j)$ for all $k \in L^j(x)$.

Assumption 3.5 (Relationship Between Candidates' Strategy Sets)**.** If candidate i's policy $y_i^{l_i^*(x)}$ loses to the policy x_j then, for each $k \in K^*(x_j)$, candidate j's policy y_j^k beats x_i.

This assumption depends on the neighborhoods only to the extent that voters who are not indifferent between x_i and x_j treat policies in the two neighborhoods the same way. Hence if it holds for some pair of neighborhoods then it holds for all smaller neighborhoods.

Example 3.6. We illustrate the assumptions with an example. There are 4 voters and the dimension of the police space is 3. So $K = 4$ and $N = 3$. Voter k's utility function is $u_k(x) = -\sum_{n=1}^{3}(x_n - a_n^k)^2$, where the ideal points are $a^1 = (1, 0, 0)$, $a^2 = (0, 1, 0)$, $a^3 = (0, 0, 0)$, and $a^4 = (0, 0, 1)$. The space of policies P is the tetrahedron obtained as the convex hull in $\mathbb{R}^3$ of the ideal points. Observe that this is the set of Pareto optimal policies, which we assume to be the set of strategies for both candidates. The following voters' weights satisfy Assumption 3.3: $w_1 = 0.11$, $w_2 = 0.20$, $w_3 = 0.29$ and $w_4 = 0.40$. For a given policy $x_i \in P$, the minimal subset L of voters for which x_i is Pareto optimal is given by the voters whose ideal points span the face on which x_i lies. For instance, for $a, b > 0$ with $a + b < 1$, $L = \{1, 2\}$ if $x_i = (a, 1-a, 0)$; $L = \{1, 3\}$ if $x_i = (a, 0, 0)$; $L = \{1, 2, 4\}$ if $x_i = (a, b, 0)$; and $L = \{1, 2, 3, 4\}$ if x_i is in the interior of P. $\overline{K}(x_i) = \{k\}$ if $x_i = a^k$ and it is empty for policies not equal to an ideal point. Assumption 3.4 is easily verified: if $x_i = a^k$ then $L = \{k\}$ and moving to the interior of P we find the required y_i (denoted y_i^k); if x_i belongs to the face spanned by voters including voter k, then the required y_i^k for voter k is found by moving to the interior of P away from a^k.

Consider $x = (x_i, x_j) \in D$ given by $x_i = (1/4, 1/4, 0)$ and $x_j = (1/4, 0, 1/4)$. Then $L^i(x) = \{2\}$, $L^j(x) = \{4\}$ and $L^0(x) = \{1, 3\}$. Also $K^*(x_i) = \{1, 2, 3\}$ and $K^*(x_j) = \{1, 3, 4\}$, so $L_i^*(x) = L_j^*(x) = \{1, 3\}$. Hence $l_i^*(x) = l_j^*(x) = \{1\}$.

Observe that $y_i^{l_i^*(x)}$ loses voter 1 (because $l_i^*(x) = \{1\}$), wins voter 3 and does not change the other two voters (relative to x_j – so 2 still prefers i's policy $y_i^{l_i^*(x)}$ over x_j, whereas 4 prefers x_j over $y_i^{l_i^*(x)}$). Given the weights specified above, $y_i^{l_i^*(x)}$ loses to x_j, as it gets $0.20 + 0.29 = 0.49$ votes, whereas x_j gets 0.51 votes. To illustrate Assumption 3.5, we must show that y_j^k beats x_i for $k = 1, 3, 4$. It is obvious for $k = 4$, as $u_4(x_j) > u_4(x_i)$, $V(x_j)$ is chosen so that this inequality is preserved for all $y_j \in V(x_j)$, and y_j^4 wins the tied voters 1 and 3. For $k = 3$, y_j^3 loses voter 3, wins voter 1 and does not change the other two voters (relative to x_i), so it gets $0.11 + 0.40 = 0.51$ votes and beats x_i. Likewise for $k = 1$, as now y_j^1 wins voter 3, so it gets 0.29 votes on top of the 0.40 votes already obtained from voter 4.

3.4. **Discussion of the Assumptions.** We now discuss the restrictiveness of the assumptions about voters' preferences. We show later that the Colonel Blotto games addressed in Section 4 necessarily satisfy Assumptions 3.4 and 3.5. In these games the voters' utilities are linear and the policy spaces are polytopes of dimension $N = K - 1$, where K is the number of voters.

In fact, if $N \geqslant K - 1$, P is convex, and voters' utility functions are differentiable, strictly quasi-concave, and generic in the space of such preferences, then the assumptions are satisfied. By genericity, the rank of the matrix of gradients at a Pareto optimal policy is $K - 1$ and Assumption 3.4 is satisfied. If x_i is an ideal policy of a voter k (and then the only voter, because of the rank condition on the matrix of gradients), then $\overline{L}_i(x)$ is nonempty iff $x_i = x_j$ and then Assumption 3.5 holds vacuously since y_i^k beats x_j. On the other hand if $L_i^*(x)$ has at least two voters, and $y_i^{l_i^*(x)}$ loses to x_j, then it must be that candidate i needs all the voters in $L^0(x)$ to win: $y_i^{l_i^*(x)}$ loses for the tied voter $l_i^*(x) \in L_i^*(x)$ with the least vote and wins for all other tied voters, and yet it loses to x_j, so candidate j needs just one of the tied voters to win election. And, for each $k \in K^*(x_j)$, it is simple to find a y_j^k that achieves that much, given that Assumption 3.4 holds, so Assumption 3.5 is satisfied too.

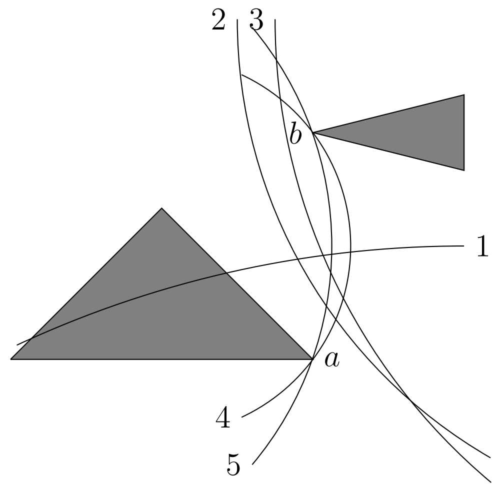
For example, if the voters have Euclidean preferences, say $u_k(p) = -\|p - a^k\|$, where $a^k \in \mathbb{R}^N$ is voter k's ideal policy, then the Pareto set is the convex hull of the ideal points a^k. If $N \geqslant K - 1$ and the ideal policies are the extreme points of the Pareto set, then the assumptions are satisfied if each candidate's strategy set is this Pareto set. But if one adds more voters with ideal points in the interior of this Pareto set then Assumption 3.4 need not be satisfied.

Example 3.7. Payoff-approachability can also fail when $N < K - 1$, even if an equilibrium and therefore the value exists. For the example, suppose there are seven voters and the policy space of each candidate is the set of lotteries over three outcomes $o_1, o_2, o_3 \in P$, so $K = 7$ and $N = 2$. The seven voters' utilities $(u_k(o_1), u_k(o_2), u_k(o_3))$ for the three outcomes are $(1, 0.6, 0)$, $(1, 0.5, 0)$, $(1, 0, 0.6)$, $(0, 1, 0.6)$, $(0.6, 1, 0)$, $(0.6, 0, 1)$, and $(0, 0.6, 1)$, and for each voter his expected utility is linear, $u_k(p) = \sum_\ell u_k(o_\ell) p_\ell$. Payoff-approachability is violated at the profile where both candidates offer the policy that yields the boundary point o_1 for sure. Even so, that profile is an equilibrium.

The following examples go further and illustrate that existence of a value is not guaranteed even when the number of voters K is small relative to the dimension N of the policy space.

Example 3.8. Refer to Figure 2 below. The strategy sets are the same and equal to the union of the two shaded triangles (the policy space P is a larger underlying set in $\mathbb{R}^2$.) There are 5 voters, each with convex preferences illustrated by the indifference curves drawn. The key feature is that feasibility considerations preclude some policies that the voters have preferences for. That is, preferences are defined over P, but only the two depicted triangles of policies are feasible. Observe that policy a is (simple) majority preferred to every feasible policy x other than policy b. Observe also that b has the upper hand in a tie with a: it gets two votes (from voters 2 and 3) whereas a gets only one (from voter 1), while voters 4 and 5 are indifferent. Under the standard tie-breaking rule under which each indifferent voters tosses a coin to decide to whom to vote, b beats a with probability 3/4 (and is beaten with probability 1/4), so $\pi(a, b) = -1/2$. Assume that voter 1's preferences are such that b is majority beaten by every policy other than a and every policy in the southwest triangle (other than a) majority beats the northeast triangle.

Figure 2



We claim the the game has no value. First, because it is a symmetric game, $\overline{v}(\pi) \geqslant 0$. Now fix σ_1. If $\sigma_1(\{a\}) \leqslant \frac{11}{24}$, then consider a sequence $\{x^n\}$ converging to a along the downward sloping edge of the southwest triangle. Each x^n beats almost every other policy: it is just beaten by a small region in the southwest triangle below voter 1's indifference curve through x^n. As this region shrinks to a, we have

$$\lim_n \pi(\sigma_1, x^n) = -\sigma_1(P \setminus \{a\}) + \sigma_1(\{a\}) \leqslant -\frac{2}{24},$$

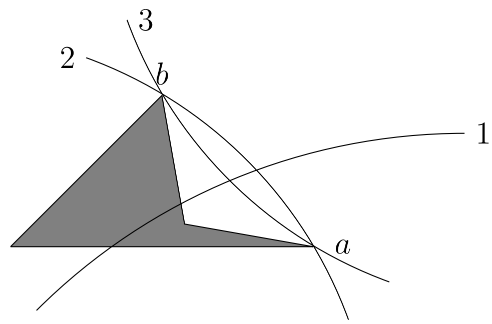
so there is x_2 with $\pi(\sigma_1, x_2) < -\frac{1}{24}$. If instead $\sigma_1(\{a\}) > \frac{11}{24}$, and $\sigma_1(P \setminus \{a, b\}) \leqslant \frac{5}{24}$, then $\pi(\sigma_1, b) < \frac{5}{24} - \frac{1}{2}(\frac{11}{24}) = -\frac{1}{48}$; otherwise, $\sigma_1(P \setminus \{a, b\}) > \frac{5}{24}$ and $\pi(\sigma_1, a) < \frac{1}{2}(\frac{16}{48}) - \frac{5}{24} = -\frac{1}{24}$. In sum, for each σ_1, there is an x_2 such that $\pi(\sigma_1, x_2) < -\frac{1}{48}$, which means that $\underline{v}(\pi) \leqslant -\frac{1}{48}$, verifying our claim.

Observe that a non generic feature of the example is that voters 4 and 5 consider two specific policies (a and b) indifferent. Still, consider a weighted majority version with only three voters: voters 1, 2 and 4, with weights $w_1 = w_4 = .3$ and $w_2 = .4$, with the same policy space P. Then again b has the upper hand in a tie with a, and a similar argument establishes that the value does not exist.

Here's another example of nonexistence of a value, now with a connected policy space.

Example 3.9. Refer to Figure 3 below. Again $X_1 = X_2$, given by the shaded area, whereas P is a larger set in $\mathbb{R}^2$. There are three voters with the indifference curves as drawn.

Figure 3



Assume that voters abstain when indifferent, and that policy b enjoys the status of a "status quo": if b is a proposed policy and a majority is not reached to upset b, then b wins. In particular, $\pi(a, b) = -1$, as only voter 1 votes for a and the other two abstain. A similar analysis shows that the game has no value (in fact, the same argument now establishes that $-\frac{1}{24}$ is an upper bound for $\underline{v}(\pi)$.)

In fact, it is simple to verify that the value does not exist in Examples 3.8 and 3.9 as long as b has the upper hand in a tie with a. If instead $\pi(a, b) \geqslant 0$, then the value is zero and both games have equilibria: if $\pi(a, b) > 0$ the unique equilibrium is to play a with probability 1 and if $\pi(a, b) = 0$ there are multiple equilibria where candidates mix between a and b, provided that the probability of a is at least 1/2. Observe that Assumption 3.4 is violated in both examples at the policy a.

Thus, Examples 3.8 and 3.9 show that a value may fail to exist even with $N = 2$ and $K = 3$ (with standard tie-breaking rule and weighted majority in Example 3.8 and with a non-standard tie-breaking rule in Example 3.9.) In such cases a general existence theorem for the value (and hence for an equilibrium) is not possible.

3.5. **Existence of an Equilibrium.** We now provide a tie-breaking rule that in combination with the previous assumptions on voters' preferences implies payoff approachability, and therefore an existence theorem for weighted-majority games. The tie-breaking rule is specified in terms of the implied payoff function $\tilde{\pi} \in \Pi$.

Definition 3.10 (Tie-Breaking Rule $\mathcal{T}$)**.** Suppose the profile x is in D.

($T1$) For each i, let $V(x_i)$ be as in Assumption 3.5. If for some i, $\overline{L}_i(x)$ is nonempty or $L_i^*(x)$ has at least two voters, and if $y_i^{l_i^*(x)}$ loses to x_j, then $\tilde{\pi}_i(x_i, x_j) = -1$ and $\tilde{\pi}_j(x_i, x_j) = +1$.

$(T2)$ In all other cases, $\tilde{\pi}_i(x_i, x_j) = 0$ for each candidate i.[20]

As above, when $y_i^{l_i^*(x)}$ loses to x_j, candidate j is in a very advantageous situation. For instance, at the pair (x_i, x_j) described in Example 3.6, candidate j has 0.40 votes already, and capturing any of the two tied voters (1 and 3) would suffice for j to win the election, whereas candidate i has to get the votes from both voters 1 and 3 to win the election. In such situations, the tie-breaking rule $\mathcal{T}$ awards the election to j. This tie-breaking rule has the following convenient property.

Lemma 3.11. *The payoff function $\tilde{\pi}$ induced by rule $\mathcal{T}$ satisfies condition (1) of Proposition 2.12.*

Proof. We can partition $D(x_i)$ into a finite number of subsets, each indexed by a triple (L^0, L^1, L^2) where, as above, candidate i gets the votes of L^i and there are ties in L^0. These sets are further decomposed by whether $(T1)$ or $(T2)$ applies, which proves (1a). To prove property (1b), fix a closed subset A^L of one of the elements of this partition with index (L^0, L^1, L^2). Then there exists $\varepsilon > 0$ such that for each $x_j \in A^L$, $|u_k(x_i) - u_k(x_j)| > \varepsilon$ for all $k \notin L^0$. Choose a ball V around x_i such that $|u_k(x_i) - u_k(y_i)| < \varepsilon$ for all $y_i \in V$. Then $\tilde{\pi}(y_i, \cdot)$ is constant on A^L for an open and dense subset of V, i.e. those y_i's for which $u_k(x_i) \neq u_k(y_i)$ for all $k \in L^0$, which verifies condition (1b). □

Now we prove the main existence theorem for weighted-majority games.

Theorem 3.12. *The game $G(\tilde{\pi})$ has an equilibrium and its value is the same as the value of $G(\pi')$ for all $\pi' \in \Pi$.*

Proof. We check that $\tilde{\pi}$ satisfies payoff approachability for an arbitrary profile (x_i, σ_j) and then apply Theorem 2.9.

By the above lemma and Proposition 2.12, we can assume that σ_j has finite support, say $x_j^1, \ldots, x_j^n$. Choose $\bar{\varepsilon} > 0$ such that for all x_j^l in the support of σ_j, and each k, $|u_k(x_i) - u_k(x_j^l)| > \bar{\varepsilon}$ if $u_k(x_i) \neq u_k(x_j^l)$. Fix a neighborhood $V(x_i)$ of x_i such that $|u_k(x_i) - u_k(x_i')| < \bar{\varepsilon}$ for all $x_i' \in V(x_i)$. By our choice of $\bar{\varepsilon}$, $V(x_i)$ is one of the neighborhoods that could be used in defining the tie-breaking rule. (In particular, for each k there are no ties between y_i^k and the x_j^l's and the former is a point of continuity against σ_j.) We show that there exists some k such that $\tilde{\pi}_i(y_i^k, \sigma_j) \geqslant \tilde{\pi}_i(x_i, \sigma_j)$, which proves payoff approachability.

For each $k \in K^*(x_i)$, $\tilde{\pi}_i(y_i^k, x_j) = \tilde{\pi}_i(x_i, x_j) = 1$ if $(T1)$ resolves the tie between x_i and x_j in favor of i, and $\tilde{\pi}_i(y_i^k, x_j) \geqslant -1 = \tilde{\pi}_i(x_i, x_j)$ if $(T1)$ resolves the ties in favor of j. Therefore,

[20] We could also use fair coin tosses for each tied voter.

if $(T1)$ applies to every x_j^l then we are done. Otherwise, let $\hat{X}_j$ be the set of x_j such that $(T2)$ applies to (x_i, x_j) and let $\hat{\sigma}_j$ be the conditional distribution over $\hat{X}_j$. We now show that there exists k such that $\tilde{\pi}_i(y_i^k, \hat{\sigma}_j) \geqslant 0 = \tilde{\pi}_i(x_i, \hat{\sigma}_j)$, which finishes the proof.

If $\overline{K}(x_i)$ is nonempty then $\tilde{\pi}_i(y_i^k, x_j) = 1$ for each $k \in \overline{K}(x)$ and each x_j in $\hat{X}_j$: indeed this is obviously true if $\overline{L}_i(x_i, x_j) = \emptyset$ since y_i^k would win each of the ties in L^0; if $\overline{L}_i(x_i, x_j)$ is nonempty, this is true since otherwise $(T1)$ applies. Therefore, we are done in this case.

Suppose $\overline{K}(x_i)$ is empty. Consider the policy $y_i \equiv y_i^{k^*}$, where k^* minimizes w_k over $K^*(x_i)$. If $k^* \notin L_i^*(x_i, x_j)$ for some x_j in $\hat{X}_j$, then obviously $\tilde{\pi}_i(y_i, x_j) = +1$; if $k^* \in L_i^*(x_i, x_j)$ and $L_i^*(x_i, x_j)$ has at least two voters, then too $\tilde{\pi}_i(y_i, x_j) = +1$, since $(T1)$ would apply otherwise. Thus among the policies in $\hat{X}_j$, y_i beats every x_j^l except, possibly, the subset $\hat{A}_j$ of those x_j's in $\hat{X}_j$ for which $L_i^*(x_i, x_j)$ is just the singleton k^*. If the probability of this subset under $\hat{\sigma}_j$ is no more than half, then $\tilde{\pi}_i(y_i, \hat{\sigma}_j) \geqslant 0$ and we are done.

Finally, suppose that the probability of $\hat{A}_j$ under $\hat{\sigma}_j$ is greater than half. Observe that $K^*(x_i)$ contains some other voter, say $\tilde{k}$, since we have assumed that $\overline{K}(x_i)$ is empty. As we argued above, for any $k \in K^*(x_i)$, y_i^k beats any x_j that x_i beats under $(T1)$ and does at least as well when x_i loses because of $(T1)$. On the set $\hat{X}_j$ we now have that $y_i^{\tilde{k}}$ beats every policy in $\hat{A}_j$, which has a probability at least half, and thus it gets a weakly higher payoff against $\hat{\sigma}_j$ than x_i, which completes the proof. □

Remark 3.13.

(1) Suppose X_i includes policies that are not Pareto optimal in X_i. Let X_i^* be the set of Pareto optimal policies in X_i. If our assumptions hold on the sets X_i^* then our results apply to obtain existence of a value over $X_1^* \times X_2^*$. We could specify payoffs at ties involving non-optimal points to extend this to an equilibrium over the bigger strategy space. But even simpler, the game over X inherits the value from the game over X^*: for each ε, our perturbation technique yields for each player i an ε-optimal strategy σ_i^ε that assigns zero probability to each indifference curve of any voter—indeed, this follows if we use for the restricted strategy sets, the sets identified by the proof of Lemma 2.4 which have the property that each element of these sets assigns zero probability to cross-sections. The same strategy is ε-optimal in X.

(2) A key feature of the tie-breaking rule $\mathcal{T}$ is $(T1)$. When it is invoked to resolve a tie between x_i and x_j, it ensures that each player can achieve the payoff from the tie by all choices of the form y_i^k. Indeed, if it is resolved in i's favor, it is guaranteed by Assumption 3.5. On the other hand, if it is resolved against i, then

it is obvious. The assumption that there are no draws (Assumption 3.3) means that $\pi_i(y_i^k, x_j) \neq 0$. But complications can arise if we allow this possibility. For simplicity suppose $L_i^*(x) = L_j^*(x) = L^0(x)$ and this set contains two voters with unequal weights. It could be that $\pi_i(y_i^{l_i^*(x)}, x_j) = 0$ for i but $\pi_i(y_i^k, x_j) = -1$ for the other voter k in $L_i^*(x)$. Thus, if we set $\pi_i(x_i, x_j) > -1$, the strategy y_i^k cannot guarantee this payoff. On the other hand if $\pi_i(x_i, x_j) = -1$, then j cannot guarantee payoff $+1$ with the strategy y_j^k. The problem here is the combination of the possibility that the game could end in a draw (each candidate gets half of the votes) with the fact that it is a weighted-majority game.

3.6. **Simple-Majority Games.** For the case of simple-majority games we specify a slightly different tie-breaking rule that implies the same result even if the number of voters is even. We use the notation from the previous subsection, except that each $w_k = 1/K$.

Obviously, Assumption 3.3 cannot hold when the number of voters is even, so it is dropped. Assumption 3.4 on diversity of preferences remains the same. Assumption 3.5 relating strategy sets has to be changed. In the following assumption and definition, we retain the notation from the previous subsection.

Assumption 3.14 (Relationship Between Candidates' Strategy Sets—The Simple-Majority Version)**.** Fix $x = (x_i, x_j) \in D$.

(1) If $\overline{L}_i(x)$ is nonempty then $|L^0(x)| \geqslant 2$.
(2) If $L_i^*(x)$ is nonempty and $|L^0(x)| \geqslant 2$ then:
   (a) If $\pi_i(y_i^k, x_j) = 0$ for some $k \in L_i^*(x)$, then for all $k \in K^*(x_j)$, $\pi_j(y_j^k, x_i) \in \{0, 1\}$ and in fact equals $+1$ if $|L^0(x)| \geqslant 3$.
   (b) If $\pi_i(y_i^k, x_j) = -1$ for some $k \in L_i^*(x)$, then for all $k \in K^*(x_j)$, $\pi_j(y_j^k, x_i) = +1$.[21]

Example 3.15.
1. In the setting of Example 3.6, set the weights to $w_k = 1/4$ for every k. Condition (1) of Assumption 3.14 holds because $\overline{L}_i(x)$ is nonempty iff $x_i = x_j$ and then $|L^0(x)| = K \geqslant 3$. Condition (2)(a) is illustrated by the policy pair (x_i, x_j) described in Example 3.6: in fact $\pi_i(y_i^k, x_j) = 0$ for $k = 1, 3$, as y_i^1 (resp. y_i^3) wins voter 3 (resp. 1) and loses voter 1 (resp. 3), so each such policy gets 2/4 votes against x_j. So we must show that $\pi_j(y_j^k, x_i) \geqslant 0$ for $k = 1, 3, 4$. And this is true, as it is equal to zero for $k = 1, 3$ (both y_j^1 and y_j^3 win one and lose one of the tied voters, so each gets 2/4 votes against x_i) and it is equal to $+1$ for $k = 4$, as y_j^4 wins both tied voters 1 and 3 and retains voter 4, so j gets 3/4 votes against x_i.

[21]Observe that if $\pi_i(y_i^k, x_j) = 0$ (resp. $\pi_i(y_i^k, x_j) = -1$) for some $k \in L_i^*(x)$, then $\pi_i(y_i^k, x_j) = 0$ (resp. $\pi_i(y_i^k, x_j) = -1$) for all $k \in L_i^*(x)$.

2. To illustrate the second part of condition (2)(a), modify Example 3.6 by adding two voters and two dimensions, $K = 6$, $N = 5$, continuing with Euclidean preferences having ideal points $a^1 = (1, 0, 0, 0, 0)$, $a^2 = (0, 1, 0, 0, 0)$, $a^3 = (0, 0, 0, 0, 0)$, $a^4 = (0, 0, 1, 0, 0)$, $a^5 = (0, 0, 0, 1, 0)$ and $a^6 = (0, 0, 1, 0, 1)$. Again the strategy sets are the Pareto set, the convex hull of the ideal policies. For simple majority rule, the weights are $w_k = 1/6$ for every k. Consider $x_i = (1/4, 1/4, 0, 0, 0)$ and $x_j = (1/4, 0, 1/4, 0, 0)$. Now $L^0(x) = \{1, 3, 5\}$, $L^i(x) = \{2\}$ and $L^j(x) = \{4, 6\}$. We have $\pi_i(y_i^k, x_j) = 0$ for $k \in \{1, 3\} = L_i^*(x)$, as y_i^1 (resp. y_i^3) wins voters 3 and 5 (resp. 1 and 5) and loses voter 1 (resp. 3), totaling 3/6 votes from voters 2, 3 and 5 (resp. 1, 2 and 5). We must show that $\pi_j(y_j^k, x_i) = +1$ for $k = 1, 3, 4$, and this follows because y_j^k for $k = 1, 3, 4$, wins at least two of the tied voters and retains voters 4 and 6 (relative to x_i), so j gets at least 4/6 votes.

3. To illustrate condition (2)(b) of Assumption 3.14, again modify Example 3.6, but now add only one voter and one dimension ($K = 5$, $N = 4$), with ideal policies $a^1 = (1, 0, 0, 0)$, $a^2 = (0, 1, 0, 0)$, $a^3 = (0, 0, 0, 0)$, $a^4 = (0, 0, 1, 0)$, and $a^5 = (0, 0, 1, 1)$, and $w_k = 1/5$ for all k. For the pair $x_i = (1/4, 1/4, 0, 0)$ and $x_j = (1/4, 0, 1/4, 0)$, we have $L^0(x) = \{1, 3\}$, $L^i(x) = \{2\}$, and $L^j(x) = \{4, 5\}$. Now $\pi_i(y_i^k, x_j) = -1$ for $k = 1, 2$, for the same reason as above, as x_j retains voters 4 and 5 and wins one more voter (voter 1 for $k = 1$ and voter 3 for $k = 3$), so it gets 3/5 votes relative to y_i^k. So we have to verify that $\pi_j(y_j^k, x_i) = +1$ for $k = 1, 3, 4$. This follows, as y_j^k for $k = 1, 3, 4$ wins at least one voter, plus voters 4 and 5 that are already won (relative to x_i).

Again, the tie-breaking rule is specified in terms of the implied payoff function $\tilde{\pi} \in \Pi$.

Definition 3.16 (Modified Tie-Breaking Rule $\mathcal{T}^S$)**.** Suppose $x \in D$.

- $(T1)$ For each i, let $V(x_i)$ be as in Assumption 3.14. Suppose for some i, $L_i^*(x)$ is nonempty and $L^0(x)$ has at least two voters. For this i:
  - (a) If $\tilde{\pi}_i(y_i^k, x_j) = 0$ for some $k \in L_i^*(x)$, then $\tilde{\pi}_i(x_i, x_j)$ is zero if $|L^0(x)| = 2$ and -1 if $|L^0(x)| \geqslant 3$.
  - (b) If $\tilde{\pi}_i(y_i^k, x_j) = -1$ for some $k \in L_i^*(x)$, then $\tilde{\pi}_i(x_i, x_j) = -1$.
- $(T2)$ Suppose $L_i^*(x)$ is empty for each i or $L^0(x) = \{ k \}$ for some k. If $\sum_{k' \in L^j(x)} w_{k'} = 1/2$, then $\tilde{\pi}_i(x_i, x_j) = -1/2$.
- $(T3)$ In all other cases, $\tilde{\pi}_i(x_i, x_j) = 0$ for each i.[22]

The rule $\mathcal{T}^S$ differs from the rule $\mathcal{T}$ used in the previous subsection only in that provisions $(T1)$(a) and $(T2)$ are added—and the condition that $L_i^*(x)$ has at least two voters if $\overline{L}_i(x)$ is

[22]Again, we could use fair coin tosses for each tied voter.

empty, when invoking $(T1)$, is relaxed—to accommodate the fact that with an even number of voters the game could end in a draw. Without these changes, $\mathcal{T}^S$ is the same as $\mathcal{T}$.

From Example 3.15(3) we see that provision $(T1)$(b) is analogous to provision $(T1)$ of tie-breaking rule $\mathcal{T}$: candidate j is in a very advantageous situation when $\tilde{\pi}_i(y_i^k, x_j) = -1$ for all $k \in L_i^*(x)$, as winning a single one of the tied voters guarantees a victory, whereas candidate i has to win all of the tied voters. In such a situation, $\mathcal{T}^S$ awards the election to j. Provision $(T1)$(a) handles draws: from Example 3.15(1), we see that $\pi_i(y_i^k, x_j) = 0$ and $|L^0(x)| = 2$ for all $k \in L_i^*(x)$ is a symmetric situation, so the rule $\mathcal{T}^S$ declares it a draw; from Example 3.15(2) we see that candidate j is in an advantageous situation when $\pi_i(y_i^k, x_j) = 0$ and $|L^0(x)| \geqslant 3$ for all $k \in L_i^*(x)$, as j has the upper hand in the non-tied battles, so $\mathcal{T}^S$ awards the election to j.

Example 3.17. Return to the setting of Example 3.15(1). Consider the pair (x_i, x_j) with $x_i = (0, 0, 0)$ and x_j in the intersection of 1's indifference surface and the face spanned by voters 1, 2 and 4, in such a way that voter 4 prefers x_j to x_i (for instance, $x_j = (\frac{3-\sqrt{5}}{4}, \frac{1}{2}, \frac{\sqrt{5}-1}{4})$). Then $L^0(x) = \{1\}$ and $L^j(x) = \{2, 4\}$, so the premise of condition $(T2)$ of the rule $\mathcal{T}^S$ applies, and the rule then says that $\tilde{\pi}_i(x_i, x_j) = -1/2$. We see that candidate j is in a stronger position because he has already secured 2/4 votes. But y_j^1 loses voter 1, so it fails to beat x_i. The relatively stronger position of candidate j is then captured by awarding the election to him with probability 3/4 rather than 1/2.

The payoff function $\tilde{\pi}$ induced by the tie-breaking rule $\mathcal{T}^S$ satisfies payoff approachability. As in the proof of Theorem 3.12, one shows that the payoff function satisfies condition 1 of Proposition 2.12 and then it is sufficient to show that payoff approachability is satisfied at each (x_i, σ_j) where σ_j has finite support in $D(x_i)$. This property is verified by Lemma B.1 in Appendix B, which then proves the existence theorem for simple-majority games.

Theorem 3.18. *The game $G(\tilde{\pi})$ has an equilibrium and its value is the value of every variant $G(\pi')$ with $\pi' \in \Pi$.*

3.7. **The Case of "Many" Voters.** When the set of voters is finite and small relative to the dimension of the policy space (i.e., $K \leq N + 1$), the conditions in the preceding subsections are typically satisfied in the space of preferences. Here we consider the other extreme: we show that when there are a continuum of voters, the Downsian model has a value for typical preferences of the voters.

Assumption 3.19. The policy space P is a ball in $\mathbb{R}^N$. The space of voters K is a compact and connected metric space. The distribution of voters is given by a probability measure λ

with the property that there exists a constant $\eta > 0$ such that for each $\delta > 0$ and each δ-ball B_δ, $\lambda(B_\delta) \geqslant \eta\delta$.[23]

Let $u : P \times K \to \mathbb{R}$ be the description of the preferences of the voters, i.e., $u(p, k)$ is the utility to voter k from the policy p. For simplicity we consider strictly quasi-concave utilities. Later, we indicate how the proofs are to be modified for the case of linear preferences.

Assumption 3.20. The utility function $u(p, k)$ is jointly continuous in (p, k). For each k, $u(\cdot, k)$ is a differentiable, strictly quasiconcave function in a closed and convex neighborhood Q of P. For each p, $\nabla_p u(p, k)$ is continuous in k. For each $p \neq q$, $\lambda(\{\, k \in K \mid u(p, k) = u(q, k) \,\}) = 0$

Remark 3.21. A canonical example of our set up can be described as follows. The policy space is the ball in $\mathbb{R}^N$ and voters have Euclidean preferences. Identifying voters by their ideal points, the voter space is either the ball itself or its boundary. Finally, the distribution λ has a strictly positive density.

Define $f : P \times P \to [0, 1]$ by $f(p, q) = \lambda(\{\, k \in K \mid u(p, k) > u(q, k) \,\})$ if $p \neq q$ and equals $1/2$ otherwise.

Assumption 3.22. f is differentiable at all (p, q) with $p \neq q$ and $f(p, q) = 1/2$.

Remark 3.23. In the case of the Euclidean preferences, the differentiability derives from λ having a strictly positive density.

Since we have assumed that the utilities are defined in some convex neighborhood Q of P as well, and that the preferences continue to be stritly quasi-concave over Q, the function f extends to a map over Q as well. Given $p \in P$ and a vector r in the unit sphere in $\mathbb{R}^N$, we denote by $L(p, r)$ the set of all points of the form $p + tr \in Q$ for $t \in \mathbb{R}$. Given a line segment L in Q through a point p, we say that p is a *median point on L* if $f(p, q) \geqslant 1/2$ for all $q \in L \setminus \{\, p \,\}$. It is a unique median point if the inequalities are strict for all q. Median points can be computed as follows. For each line $L(p, r)$ and t such that $p + tr \in Q$, define $g(t, p, r)$ to be the measure of the set of voters whose ideal points on $L(p, r)$ lie at points $p + t'r$ for $t' \leqslant t$. Then p is a median point of $p + tr$ iff $g(0, p, r) = 1/2$.

Remark 3.24. In the example of Euclidean preferences, median points can be described geometrically as follows. Given a line $L(p, r)$ Let $H_{L(p,r)}$ be the hyperplane with normal r and intersecting the line at p. Then p is a median point on $L(p, r)$ iff the measure of voters

[23]When K is a subset of a Euclidean space, this last assumption holds if λ has a strictly positive density w.r.t. the Lebesgue measure.

on both half spaces is exactly one half. Moreover, median points are unique, since λ has a positive density.

Assumption 3.25. For each $p \in P$ and unit vector r, $g(\cdot, p, r)$ is continuous and strictly increasing in t whenever $g(t, p, r) \in (0, 1)$. Moreover for a.e. r, $g(0, p, r) \neq 1/2$, i.e., p is not a median point on $L(p, r)$; if $p \in \partial P$, then this is also true for a.e. r in the tangent space to P at p.

Remark 3.26. Assumption 3.25 is a regularity assumption. Going back to the Euclidean case, since λ assigns zero measure to hyperplanes and has a strictly positive density, the monotoncity assumption is satisfied. Also, for the same reasons, for each r there is a unique $p(r)$ such that such that $g(0, p(r), r) = 1/2$. Moreover this is a differentiable function of r. For p not to be a median for a.e. r is to say that the derivative of $p(r)$ w.r.t. r is not zero. This last point is implied by a regularity argument since the map $p(r)$ sends the unit sphere in $\mathbb{R}^{N-1}$ to P, which is N-dimensional.

For simplicity, we consider a symmetric game. Thus, the strategy sets are $X_1 = X_2 = P$. Let D_0 be the diagonal of X, i.e., $D_0 = (p, p) \in X$. We define the payoffs as follows. For each pair (x_1, x_2), $\pi_1(x) = 1$ (resp. -1) if $f(x)$ is greater than $1/2$ (resp. smaller than $1/2$). It is zero if $f(x) = 1/2$. That is, the tie-breaking rule used is the one that assigns zero payoff to every tie.[24] For each x_i, $D(x_i)$ is the set of x_j such that $f(x_i, x_j) = 1/2$ along with the point $x_j = x_i$. Let $G(\pi; \lambda)$ denote the normal form game with payoff π and underlying distribution λ on K.

We have a preliminary lemma about the behavior of the function f.

Lemma 3.27. *For $x_1 \neq x_2$, if $f(x_1, x_2) = 1/2$, then $\nabla_{x_1} f(x_1, x_2) \cdot (x_2 - x_1) > 0$.*

Proof. Fix $x_1 \neq x_2$ and let $\alpha : K \to \mathbb{R}$ be the function given by $\alpha(k) = u(x_1, k) - u(x_2, k)$. Let $K_0 = \alpha^{-1}(0)$. We claim first that K_0 is nonempty. Indeed, as α is continuous in k, the connectedness of K implies that its image under α is connected. Therefore, if K_0 is empty, then the image of K under is contained either in $\mathbb{R}_+$ or $\mathbb{R}_-$, i.e., $f(x_1, x_2)$ is either 0 or 1. Thus, K_0 is nonempty.

For each $k \in K_0$, strict quasiconcavity of $u(\cdot, k)$ in a neighborhood of x_1 in Q and $u(x_1, k) = u(x_2, k)$ imply $\nabla_{x_1} u(x_1, k) \cdot (x_2 - x_1) > 0$. Moreover, the ideal point of an $k \in K_0$ along the line $x_1 + t(x_2 - x_1)$ is at some $t^*(k) > 0$. By continuity of $\nabla_{x_1} u(x_1, k)$ in k and compactness of K_0, there exist $c > 0$, $t^* > 0$ and a neighborhood V_0 of K_0 such that $\nabla_{x_1} u(x_1, k) \cdot (x_2 - x_1) > 2c$ and $t(k) > t^*$ for all $k \in V_0$. Observe now that for each $k \in V_0$, $g(t, k) \equiv u(x_1 + t(x_2 -$

[24] This is the tie-breaking rule employed by Duggan [10].

$x_1), k) - u(x_1, k) > ct$ for all $0 < t \leqslant t^*$: Indeed, for each $k \in V_0$, there exists some $\bar{t}$ such that $g(t, k) > ct$ for all $0 < t < \bar{t}$; the fact that the ideal point of k occurs for $t > t^*$ implies that this inequality for all $\bar{t} \leqslant t \leqslant t^*$.

By continuity of α in k, there is $a > 0$ with $\alpha^{-1}([-a, a])$ contained in V_0, and a Lipschitz constant ζ. Observe that for $k \in V_0$ and $t \leqslant t^*$, if $\alpha(k) > -ct$ then k strictly prefers $x_1(t) \equiv x_1 + t(x_2 - x_1)$ to x_2. Now consider $x_1(t)$ for $t < 2a/c$, so that there is $k(t) \in V_0$ with $\alpha(k(t)) = -ct/2$. We have: (i) every $k \in \alpha^{-1}([0, \infty))$ prefers $x_1(t)$ to x_2; (ii) every k in the $ct/2\zeta$-ball around $k(t) \in V_0$ with $\alpha(k(t)) = -ct/2$ prefers $x_1(t)$ to x_2, provided that the ball is contained in V_0, which is true for t small enough. The measure of the first set is exactly $1/2$ since $f(x_1, x_2) = 1/2$. The measure of the second set is at least $ct\eta/2\zeta$, by Assumption 3.19. Therefore, $f(x_1(t), x_2) \geqslant 1/2 + ct\eta/2\zeta$, establishing that the directional derivative of f at (x_1, x_2) is at least $c\eta/2\zeta > 0$. □

Lemma 3.28. *Under Assumptions 3.19 and 3.20, the game $G(\pi; \lambda)$ is mildly discontinuous.*

Proof. It is sufficient to show that for each x_2, the set of $x_1 \neq x_2$ such that $f(x_1, x_2) = 1/2$ is a lower-dimensional subset of P: then since ties occur only when $f(x_1, x_2) = 1/2$, the result follows from Lemma 2.4. For each x_2, $1/2$ is a regular value of $f(\cdot, x_2)$ for each x_2, by the previous lemma. Hence, the set of x_1 that ties with x_2 is a manifold of dimension $N - 1$. □

We now verify that $G(\pi; \lambda)$ satisfies payoff approachability, and therefore has a value that is independent of the tie-breaking rule.

Theorem 3.29. *The game $G(\pi, \lambda)$ has an equilibrium and its value is the same as the value of $G(\pi', \lambda)$ for all $\pi' \in \Pi$.*

Proof. As the game is symmetric, it suffices to show that π_1 satisfies payoff approachability. Let (x_1, σ_2) be a strategy profile such that $\sigma_2(D(x_1)) = 1$. We will show that $\sup_{x_1^n \to x_1} \limsup_n \pi_1(x_1^n, \sigma_2) \geqslant 0$ for all sequences $\{ x_1^n \}$ of points of continuity of σ_2, which, in light of Lemma 2.5, shows that π_1 satisfies payoff approachability. (Recall that every stragegy σ_2 is the average of two strategies σ_2^c and σ_2^d where x_1 is continuous against σ_2^c and $\sigma_2^d(D(x_1)) = 1$.) We break up the proof into cases.

Case 1. Suppose $x_1 \notin \partial P$. For each ε, let U_ε be the ε-ball around x_1 and let σ_1^ε be the uniform distribution over U_ε. It is sufficient to prove that $\lim_{\varepsilon \to 0} \pi_1(\sigma_1^\varepsilon, \sigma_2) \geqslant 0$. To prove this, it is sufficient to show that for each $x_2 \in D(x_1)$, $\lim_{\varepsilon \to 0} \pi_1(\sigma_1^\varepsilon, x_2) \geqslant 0$. Therefore, fix $x_2 \in D(x_1)$. For each ε, σ_1^ε can be decomposed as follows. For each line segment $L = L(x_1, r)$ through x_1, let μ_L^ε be the uniform distribution over the intersection of the line segment L with U_ε, and let ν be the uniform distribution over line segments. Then σ_1^ε can be expressed as $d\sigma_1^\varepsilon(y_1) = d\nu(L)d\mu_L^\varepsilon$ where $L = L(x_1, r(y_1))$ is the line segment through x_1 containing y_1. To prove that $\lim_{\varepsilon\to 0} \pi_1(\sigma_1^\varepsilon, x_2) = 0$, it is sufficient to show that for a.e. L, $\pi_1(\mu_L^\varepsilon, x_2) = 0$. This point follows if a.e. L, $f(\cdot)$ is strictly greater than 1/2 on one side of x_1 on the line $L \cap (U_\varepsilon \setminus \{x_1\})$ and strictly less than 1/2 on the other side, for all small ε. In particular, if f is differentiable at x w.r.t. x_1, this monotonicity property is implied by a non-zero derivative of f w.r.t. x_1 along L.

Case 1a. Suppose $x_2 \neq x_1$. As we just indicated, we have to show that $\nabla_{x_1} f(x) \neq 0$. But this follows from Lemma 3.27, as the directional derivative in the direction of $(x_2 - x_1, 0)$ is positive.

Case 1b. Suppose $x_2 = x_1$. Then, for a.e. r, $g(0, x_1, r) \neq 1/2$. If $g(0, x_1, r)$ is smaller than 1/2, then x_1 beats $x_1 + tr$ for $t < 0$ while for all small $t > 0$, $x_1 + tr$ beats x_1. Likewise, if $g(0, x_1, r)$ is greater than 1/2, then x_1 beats $x_1 + tr$ for $t > 0$ and is beaten by $x_1 + tr$ for all small $t < 0$. Again, this means that along the lines $L(x_1, r)$ generated by a.e. r we have $\pi_1(\mu_L^\varepsilon, x_2) = 0$ for small enough ε.

Case 2. Suppose now that $x_1 \in \partial P$. Without loss of generality we can assume that x_1 is the origin of $\mathbb{R}^N$, that the tangent space T to P at x_1 is the half space where the N-th coordinate is zero, and P lies in the half space where the last cordinate is non-negative. Let R be the unit sphere in T. For each $r \in R$ and ε, let r_ε be the vector $(\sqrt(1+\varepsilon^2)(r, \varepsilon)$ in the unit sphere in $\mathbb{R}^N$. There exists $\varepsilon_0 > 0$ such that for each $r \in R$ and each $0 < \varepsilon \leqslant \varepsilon_0$, there is a unique $t^*(r_\varepsilon) > 0$ such that $x_1 + tr_\varepsilon$ belongs to P for all $0 \leqslant t \leqslant t^*(r_\varepsilon)$ with x_1 and $x_1 + t^*(r_\varepsilon)r_\varepsilon$ being the only boundary points on the line. For each $0 < \varepsilon < \varepsilon_0$, let R_ε be the set of vectors $x_1 + t^*(r_\varepsilon)r_\varepsilon$ and let σ_1^ε be the uniform distrition over the vectors $r_{\varepsilon'}$ for $0 \leqslant \varepsilon' \leqslant \varepsilon$ defined by taking a uniform distribution ν over lines in R_ε and a uniform distribution μ_ε over $[0, \varepsilon]$, i.e., $d\sigma_1^\varepsilon(r_{\varepsilon'}) = d\nu(L)\mu_\varepsilon(\varepsilon')$ where L is the line containing r. As in Case 1, we will show that for each x_2 in the support of σ_2, $\pi_1(\sigma_1^\varepsilon, x_2) \geqslant 0$ for all small ε. To do this, it is sufficient to show that for generic lines L in R_ε, the uniform distribuion μ_L^ε over the set $\hat{L}^\varepsilon$ consisting of $r_{\varepsilon'}$ with $r \in L$ and $0 < \varepsilon' < \varepsilon$ does at least as well as x_1 against x_2 for all small ε.

Case 2a. Suppose $x_2 \neq x_1$, as in Case 1, the directional derivative towards $x_2 - x_1$ is non-zero. If the tangent space T' in X_1 generated by $\nabla_{x_1} f(x)$ is T then since x_2 belongs to P, all points close to x_1 beat x_2 and in particular, for all small ε, points in $\hat{L}^\varepsilon \setminus x_1$ beat x_2. Hence $\pi_1(\mu_L^\varepsilon, x_2) = 1$ in this case. If the tangent space is transverse to $\mathbb{R}^{N-1}$, then as long as as L does not belong to the tangent space given by $\nabla_{x_1} f(x)$ (which would be true for generic L), then for each small $\varepsilon > 0$ r_ε belongs to one side of T' iff $(-r)_\varepsilon$ (the vector generated by $-r$) lies on the other side. Thus, Hence, for small ε, $\pi_1(\mu_L^\varepsilon, x_2) = 0$.

Case 2b. Suppose $x_1 = x_2$. Then for a.e. lines $L(x_1, r)$ for $r \in T$, $g(0,0,r) \neq 1/2$. If it is greater than $1/2$, by continuity of g in r, $g(0,0,r_\varepsilon) > 1/2$ for all small ε, i.e., r_ε beats x_2 for all such ε. Since either $g(0,0,r)$ or $g(0,0,-r)$ is greater than $1/2$ for each r, for all small ε, r_ε either beats x_2 or $(-r)_\varepsilon$ does. Thus, $\pi_1(\mu_L^\varepsilon, x_2) \geqslant 0$. □

Remark 3.30. We now sketch the argument in the case where the utilities of the voters are all linear. Each voter has a unique ideal point in P, which is on the boundary, and we parametrize voters by their ideal points. We assume that the measure λ has a strictly positive density on $\partial P \equiv K$. Assumption 3.20 holds because λ has a positive density. Assumption 3.22 is seen to hold as is the condition that some voter is indifferent between p and q. Since for any line L, voters have extreme preferences (preferring one or the other extreme point), the median point is unique unless equal measures of voters prefer one to the other. Typically we have Assumption 3.25 holding as well. (Note: it does not hold if the distribution λ is uniform over the ball!) Lemma 3.27 is not true anymore. But, it can be shown that the partial derivative of f w.r.t. either variable is non-zero since along generic lines through, say, x_1, the ideal points are extreme. Since the proof of the result only requires that the partials of f do not vanish, it goes through in the linear case.

3.8. **An Asymptotic Result.** The existence result of the previous section can be used to prove the existence of ε-optimal strategies for games with a large but finite number of voters. This follows from a continuity result: if a game satisfies the assumptions of that section, then for each $\varepsilon > 0$, all games with all distributions "close" to λ have an ε-equilbirium.

We use the topology of weak-$*$ convergence on the space of distributions over K, which is metrizable because K is so. In the following theorem a basic game $G(\pi; \lambda)$ is fixed satisfying Assumptions 3.19, 3.20, 3.22, and 3.25. For each distribution λ' over K, $G(\pi; \lambda')$ is the game that differs from $G(\pi; \lambda)$ only in that the distribution λ is replaced with λ'.

Theorem 3.31. *For each $\varepsilon > 0$ there exists $\delta > 0$ and σ such that for all λ' within δ of λ, σ is an ε-equilibrium of $G(\pi; \lambda')$.*

Proof. Fix $\varepsilon > 0$. Each player i has a strategy σ_i that is $\varepsilon/4$-optimal in $G(\pi; \lambda)$ and assigns zero probability to each x_i. In fact, for each integer m, let U_m be the $1/m$ ball around x_i. Cover X_i with such balls and extract a finite subcover $\{U_m^h\}$ Let μ_m^h be the uniform distribution over U_m^h. Define a game G^m for each m as follows. The strategy set of player j is Σ_j, and the strategy set of player i is the space Σ_i^m of convex combinations of μ_m^h's. The payoff function is the restriction of π to $\Sigma_i^m \times \Sigma_j$. Such payoffs are continuous and bilinear, so G^m has an equilibrium σ^m and its value v^m is equal to 0. Proceeding as in Lemma A.2, we conclude that for large m, $\pi_i(\sigma_i^m, x_j) \geqslant -\varepsilon/4$, for all $x_j \in X_j$, verifying the claim. Let

σ_1 denote the σ_1^m satisfying $\pi_1(\sigma_1, x_2) \geqslant -\varepsilon/4$ for all $x_2 \in X_2$. We now show that there is $\delta > 0$ such that σ_1 is an ε-optimal strategy for the game $G(\pi; \lambda')$ when λ' is δ-close to λ. The argument is similar for player 2.

For each $\gamma > 0$, $f^{-1}([1/2, 1/2 + \gamma]) \cup D_0$ is a closed set. Therefore, there exists $\gamma > 0$ such that for each x_2, $\sigma_1(\{x_1 \mid x_1 = x_2 \text{ or } (x_1, x_2) \in f^{-1}([1/2, 1/2 + \gamma]) < \varepsilon/8$. Therefore, for any x_2, the probability that 1 wins at least $1/2 + \gamma$ share of the votes is at least $(1/2)(1 - \varepsilon/4 - 2(\varepsilon/8)) = 1/2 - \varepsilon/4$.

Since $\sigma_1(x_1) = 0$ for each $x_1 \in X_1$, we can find an open neighborhood U_0 of the diagonal D_0 such that for each x_2, $\sigma_1(\{x_1 \mid (x_1, x_2) \in U_0\}) < \varepsilon/4$. From Assumption 3.20, for each $(x_1, x_2) \in X \setminus U_0$, $\lambda(\{k \mid u(x_1, k) = u(x_2, k)\}) = 0$. Therefore, we can find $\alpha > 0$ small enough such that the $\lambda(\{k \mid |u(x_1, k) - u(x_2, k)| \leqslant \alpha\}) < \gamma/2$.

We now have that, for each $x_2 \in X_2$, the probability under σ_1 is at least $1/2 - \varepsilon/2$ that: (1) $(x_1, x_2) \notin U_0$; (2) $\lambda(K_\alpha) \geqslant 1/2 + \gamma/2$ for $K_\alpha = \{k \mid u(x_1, k) - u(x_2, k) > \alpha\}$.

Since $u(x_1, k) - u(x_2, k)$ is continuous on the compact $(X \setminus U_0) \times K$, it has a Lipschitz constant η. Let $\delta = \min(\alpha/\eta, \gamma/2)$. Consider a distribution λ' within δ of λ. For each (x_1, x_2) satisfying conditions (1) and (2) above, the δ neighborhood K_α^δ of the set K_α is contained in the set of voters who strictly prefer x_1 to x_2; as $\lambda'(K_\alpha^\delta) \geqslant \lambda(K_\alpha) - \delta > 1/2$, x_1 would beat x_2 in the game λ' in this event. Under σ_1, the probability of this event is at least $1/2 - \varepsilon/2$, so σ_1 guarantees a payoff of at least $1/2 - \varepsilon/2 - (1/2 + \varepsilon/2) = -\varepsilon$ against every strategy of player 2. □

Now, a large finite game can be represented as a game $G(\pi; \lambda')$ with λ' with finite support. If in addition λ' is δ-close to λ, the finite game has an ε-equilibrium. Alternatively, consider a sequence of elections with increasing number of voters. If the limiting distribution λ of voters satisfies the conditions set forth above, then for any $\varepsilon > 0$ there is a large enough electorate such that the corresponding game has an ε-equilibrium. If the limiting distribution λ does not satisfy the conditions above, then there is a distribution λ' closeby that does satisfy the conditions. Moreover, such λ' will be the limiting distribution of a sequence of elections that is close to the original sequence of elections. So, again, for each $\varepsilon > 0$ there is a large enough electorate taken from the perturbed sequence of elections, such that the corresponding game has an ε-equilibrium. In all, a game with a large enough electorate can be approximated by a game with a continuum of players which has an equilibrium.

4. Majority Games of Resource Allocation

This section addresses a special case of the formulation and results in Sections 2 and 3. The two players compete for votes in several constituencies, called battlegrounds. The winner of the game is again the player who wins more votes. The key feature now is that a player wins a battle if he allocates more of his available resources to that battle than his opponent does. Thus the game is a majority-rule version of a Colonel Blotto game.[25]

4.1. **Formulation.** The game G is a weighted-majority game specified as follows. Each player i has an amount R_i of a resource that he allocates among the battles. Assume that $R_1 \geqslant R_2 > 0$ and that the number of battles is an integer $K > 2$. A pure strategy $x_i = (x_{i,k})_{k=1,\dots,K}$ for player i allocates a nonnegative amount $x_{i,k}$ of his resource to battle k. Thus his set of pure strategies is $X_i \equiv \{\, x_i \in \mathbb{R}^K_+ \mid \sum_{k=1}^K x_{i,k} = R_i \,\}$. For each profile $x \equiv (x_1, x_2) \in X_1 \times X_2 \equiv X$ of pure strategies for the two players, player i wins battle k, and the other player j loses, if $x_{i,k} > x_{j,k}$. If $x_{i,k} = x_{j,k}$ then a tie-breaking rule determines the winner of battle k.

For each battle k, the winner of the battle obtains w_k votes, where $0 < w_k < 1/2$ and $\sum_k w_k = 1$. We assume that $\sum_{k \in L} w_k \neq 1/2$ for each subset L of K, except when we consider simple-majority games.[26] Player i wins the game and gets payoff $+1$ if $\sum_{k \in W_i} w_k > 1/2$, where W_i is the set of battles he wins; similarly, he loses and gets payoff -1 if $\sum_{k \in W_i} w_k < 1/2$. The players' payoffs are both zero if both win $1/2$ votes. Thus, if there are no tied battles or the resolutions of ties are inconsequential, then a player's payoff is either $+1$ if he wins a weighted majority of votes, or -1 if he loses. If resolutions of tied battles affect the outcome of the game then his expected payoff is some number in the interval $[-1, +1]$. Either way, player i's payoff function is $\pi_i : X \to [-1, +1]$, and $\pi_1(x) + \pi_2(x) = 0$ for every profile $x \in X$.

Before proceeding, we note that our results, except those in Section 4.2 for simple-majority games, go through if we use a plurality rule, so that player i's payoff is $\sum_{k \in W_i} w_k$. This makes the game a constant-sum game that is strategically equivalent to a zero-sum game. In fact the proofs are simpler since then we can work with the standard tie-breaking rule in which the winner of each tied battle is chosen by the toss of a fair coin. For more general non-constant-sum games our basic existence theorem—which shows the existence of an equilibrium for the

[25]Duggan [11] proves existence of an equilibrium of this game for the case of simple-majority rule and symmetric resources. The other literature on Colonel Blotto games assumes that each player maximizes the number of battles won, rather than winning a majority. This case is sometimes interpreted as relevant to plurality and majority rule but the connection is not exact when the number of battles exceeds three; see Laslier [17, 18] and Laslier and Picard [19] for other comparisons. A referee informs us that the first solution to the game with three battles and symmetric resources appears in Borel and Ville [6]. This literature culminates in the article by Roberson [24], who provides a complete analysis of such games, and in Hart [14] for the case that resources are allocated in discrete amounts.

[26]This assumption is not needed if we consider the case where one player wins all ties.

game $G(\tilde{\pi})$ when it satisfies payoff approachability—goes through; such games are studied by Kvasov [16], Kvasov and Roberson [25], Roberson [24], and Thomas [28].

This model is a special case of those in Section 3. The policy space P is the union of X_1 and X_2. Each battle represents a voter whose utility function is $u_k(x_i) = x_{i,k}$.[27]

The following theorem extends a result obtained by Duggan [11], who proves existence of an equilibrium for simple-majority rule with symmetric resources and the standard tie-breaking rule.

Theorem 4.1. *If the tie-breaking rule is $\mathcal{T}$ (or $\mathcal{T}^S$ in the case of simple majority) then the game has an equilibrium that yields the value, and any other tie-breaking rule yields the same value.*

Proof. We verify the assumptions stated in Sections 3.5 and 3.6 and apply Theorems 3.12 and 3.18, respectively. Assumption 3.3 is stated in the formulation. To check the other assumptions, remark first that $K^*(x_i)$ is the set of battles whose coordinates are positive. In particular, $\overline{K}(x_i)$ is a singleton for a vertex (the voter corresponding to the battle getting all the resources) and empty elsewhere. With this feature, Assumption 3.4 is easily verified. In fact, for each coordinate that is positive, we can reduce it by an arbitrarily small amount and assign a strictly higher amount to all other battles.

Regarding Assumption 3.5, suppose x_i is tied with x_j, $L_i^*(x)$ is nonempty, with $|L_i^*(x)| \geqslant 2$ if x_i is not a vertex, and $y_i^{l_i^*(x)}$ loses to x_j. Since $L_i^*(x)$ is nonempty, if x_i is a vertex then it must be that $L^0(x)$ contains this one non-zero coordinate of player i. Moreover, $i = 2$ and $R_2 < R_1$: indeed as j must assign R_i to this battle as well, $R_j \geqslant R_i$, but if $R_j = R_i$, then $x_i = x_j$ and $y_i^{l_i^*(x)}$ would beat x_j. Since $R_j > R_i$, x_j is not a vertex, i.e. $K^*(x_j)$ has at least two nonzero coordinates. As a result, each y_j^k beats x_i on all coordinates except possibly for the one corresponding to the vertex, and thus it wins the game (recall that $w_k < 1/2$ for all k).

If x_i is not a vertex then $L_i^*(x)$ has at least two elements. When $y_i^{l_i^*(x)}$ loses to x_j it means that j could win the game by winning any of the battles in $L_i^*(x)$. Since $L_i^*(x)$ equals $L_j^*(x)$ and has at least two non-zero coordinates, every y_j^k would accomplish this as it would reduce at most one of the nonzero coordinates in $L^0(x)$.

Finally we check Assumption 3.14 for the simple-majority case (with even or odd number of battles). Suppose x_i is a vertex. If x_i ties with x_j just on the one non-zero coordinate

[27]Rather than viewing electoral competition as occurring in the space of proposed policies, as the strategy space one can equivalently use the space of voters' utility profiles generated by policies. In this framework, Colonel Blotto games are the special case in which the strategy spaces are simplices.

of x_i, x_j wins, as $K \geqslant 3$. Thus, condition (1) holds. As for condition (2), suppose $L_i^*(x)$ is nonempty and $|L^0(x)| \geqslant 2$. If $\tilde{\pi}_i(y_i^k, x_j) = 0$ for some k, then K is even and $|L^j(x)| = K/2-1$. Each strategy $y_j^{k'}$ of player j would lose battle k' if $k' \in L^0(x)$ but win every other battle in $L^0(x)$. Thus, $\tilde{\pi}_j(y_j^k, x_i) = 0$ if $y_j^k \in L^*(x)$ and $|L^0(x)| = 2$; otherwise, it equals $+1$, as required by condition (2a). If $\tilde{\pi}_i(y_i^k, x_j) = -1$, then $|L^j(x)|$ is the greatest integer not more than $K/2$. Each y_j^k can win at least one of the battles in $L^0(x)$ and thus win the war, giving us condition (2b). □

Remark 4.2. We need something stronger than the standard rule if payoff approachability is to hold. To see the problems with the standard rule, suppose $K = 3$, we have simple majority rule, $R_1 > R_2$, $i = 2$, and x_i allocates zero resources to the first battle and $R_2/2$ to each of the other two. Suppose x_j is the pure strategy of player $j = 1$ that allocates a positive amount to the first battle and ties with player i on the other two battles. Then using tosses of a fair coin for each of the ties gives player i a probability 1/4 of winning. Every nearby strategy loses.

4.2. **Existence of an Equilibrium With Zero Probability of Ties.**

The results above can be strengthened for simple-majority games. For this class of games we use the existence result from Section 3.4, under the tie-breaking rule $\mathcal{T}^S$.

Permutations of the battles induce a symmetry group, and therefore among the equilibria there are some that inherit the symmetries of the game. We show that these equilibria have zero probability of ties except for a single critical value of R_1/R_2.

Assume that $w_k = 1/K$ for all k, so that G is a simple-majority game. Thus a player winning $1 + \lfloor K/2 \rfloor$ battles wins the game.[28] Let $r^* = K/\lceil K/2 \rceil$. Diermeier and Myerson [8] call r^* the hurdle factor and prove the following.

Proposition 4.3. *If $R_1/R_2 > r^*$ then player 1 has a strategy that wins for sure independently of the tie-breaking rule.*

Sketch of Proof. The pure strategy of player 1 that allocates his resources uniformly across all the battles wins the game against every strategy of player 2, and no ties occur that could affect whether player 1 wins. □

In the most relevant case that R_1/R_2 is strictly below the hurdle factor, there exists an equilibrium in which the tie-breaking rule is invoked with zero probability, as we now verify.

[28] $\lfloor K/2 \rfloor$ is the greatest integer not more than $K/2$, and $\lceil K/2 \rceil$ is the least integer not less than $K/2$.

Because the game G uses a simple majority to decide the winner, it treats battles symmetrically.[29] Every permutation $\phi : \{1,\ldots,K\} \to \{1,\ldots,K\}$ of the battles defines a homeomorphism of X_i with itself that sends each x_i to x_i^ϕ where $x_{i,k}^\phi = x_{i,\phi(k)}$ for each k. Obviously $\pi_i(x_i,x_j) = \pi_i(x_i^\phi, x_j^\phi)$ for all ϕ. Let Φ be the set of all permutations. Define $H_i : X_i \to \Sigma_i$ by mapping each x_i to the uniform mixture over the set $\{x_i^\phi\}_{\phi\in\Phi}$. This map extends to a function from Σ_i to Σ_i. Define $\tilde{\Sigma}_i \equiv H_i(\Sigma_i)$ and let $\tilde{\Sigma} = \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$. There exists an equilibrium σ^* of the game G with the tie-breaking rule $\mathcal{T}^S$ such that $\sigma^* \in \tilde{\Sigma}$ and $\sigma_1^* = \sigma_2^*$ if $R_1 = R_2$. To see this, apply the perturbation method in the proof of Theorem 2.9 but now choosing the strategy sets to be symmetric with respect to the battles and perturbing the strategies of both players simultaneously. These perturbed games have an equilibrium that is invariant under all the symmetries of the game and hence the limit of these equilibria as the perturbations shrink inherit the same properties. The following result, proved in the Appendix, shows that ties occur with zero probability in equilibrium.[30]

Theorem 4.4. *If $R_1/R_2 < r^*$ then $(\sigma_1^* \otimes \sigma_2^*)(D) = 0$. That is, at the equilibrium σ^* the probability is zero that the tie-breaking rule $\mathcal{T}^S$ is invoked.*

Remark 4.5. In the knife-edge case that R_1/R_2 is exactly equal to the hurdle factor r^*, ties can occur in an equilibrium, and optimal strategies can depend on the tie-breaking rule. The uniform strategy described in the proof of Proposition 4.3 continues to be a maximin strategy of player 1 under rule $\mathcal{T}$, or if he wins all ties then again he can assure the value $+1$. But player 1 does not have a maximin strategy if the tie-breaking rule is the standard rule that tosses a fair coin to resolve each tied battle. For simplicity, we illustrate the case $K = 3$, $R_1 = 3/2$, $R_2 = 1$, and $r^* = 3/2$. Let π be the expected payoff function induced by the standard rule. As argued above, because of the symmetry of the battles, if player 1 has a maximin strategy then he has one that is invariant under all permutations of the coordinates. Thus fix a strategy $\tilde{\sigma}_1$ that is invariant under the symmetries of the game. For each x_i, denote the rank order by $(x_{i,k_1}, x_{i,k_2}, x_{i,k_3})$, with $x_{i,k_1} \leqslant x_{i,k_2} \leqslant x_{i,k_3}$ for distinct battles k_1,k_2,k_3. Let $\tilde{\sigma}_1(\{x_1 : x_{1,k_2} \leqslant 1/2\}) = \alpha \geqslant 0$. Observe that the probability of $x_2^1 = (1/2,1/2,0)$ winning is bounded below by $(1/6)\alpha$. Thus $\pi_1(\tilde{\sigma}_1, x_2^1) \leqslant 1 - \alpha + 5\alpha/6 = 1 - \alpha/6$. For each $b \in (1/2, 3/4]$, let $\tilde{\sigma}_1(\{x_1 : b/2 + 1/4 \leqslant x_{1,k_2} \leqslant b\}) = \beta(b) \geqslant 0$. Observe that we can find b and $\beta(b) > 0$ when $\alpha = 0$ and that $\alpha = 1$ when $\beta(b) = 0$ for all such b. Now, because for

[29]This feature is also exploited by Hart [14] for the discrete case.

[30]Zero probability of ties does not imply irrelevance of the tie-breaking rule, since it still has a role in deterring deviations from the equilibrium strategies. We conjecture (at least in the symmetric case where both candidates have equal resources, but possibly also more generally except for a single critical ratio of resources) that the game has an equilibrium that remains an equilibrium for every tie-breaking rule.

each x_1 with $b/2 + 1/4 \leqslant x_{1,k_2} \leqslant b$, we necessarily have $x_{1,k_1} \leqslant 1 - b$, the probability of the strategy $x_2^b = (b, 1-b, 0)$ winning is bounded below by $(1/6)\beta(b)$. So $\pi_1(\tilde{\sigma}_1, x_2^b) \leqslant 1 - \beta(b)/6$. Combining the two bounds, we must have $\inf_{x_2 \in X_2} \pi_1(\tilde{\sigma}_1, x_2) \leqslant \min\{1 - \alpha/6, 1 - \beta(b)/6\}$. Theorem 4.1 above ensures that the game has a value, and the value is independent of the tie-breaking rule. Because the value of the game with payoff function π^+ is $+1$ (the maximin strategy for player 1 assigns 1/2 to each battle), the value of the game with payoff function π is $+1$. So a maximin strategy $\tilde{\sigma}_1$ for player 1 must satisfy $\inf_{x_2 \in X_2} \pi_1(\tilde{\sigma}_1, x_2) = +1$. But this requires that α and $\beta(b)$ are zero for every b, which is impossible. So player 1 does not have a maximin strategy, and a Nash equilibrium cannot exist. Note that this implies that the game with payoff function π violates better-reply security even though the value exists.

5. Concluding Remarks

The absence of general theorems establishing existence of values, optimal strategies, and equilibria of zero-sum majority games has long impeded applications to electoral competition and redistributive politics. In studies of elections, reliance on one-dimensional policy spaces has limited the relevance to practical affairs, while for multidimensional policy spaces the general results show only that if an equilibrium exists then its support lies within the 'uncovered' set (cf. Banks and Duggan [1], who assume the game is symmetric). In studies of resource allocation in electoral campaigns and lobbying, the absence of general existence results has impaired conclusions about effects of asymmetries in resources available to the candidates. The technical difficulties stem from discontinuities in payoffs at ties, and therefore hinge on how ties are resolved.

Our two general results in Section 2 provide alternative tools. Theorem 2.6 shows that when all ties are resolved in favor of one player then the value exists and that player has an optimal strategy that ensures the value. This conclusion is especially useful in models of elections, where otherwise assumptions about voters' preferences are required. Theorem 2.9 shows that tie-breaking rules satisfying payoff approachability imply better-reply security and therefore equilibria exist that yield the value; and importantly, any other tie-breaking rule yields the same value, so ε-equilibria exist.

This result applies to the models of elections addressed in Section 3, where specific tie-breaking rules and either diversity of voters' preferences or regularity of voters' preferences (for large electorates) imply payoff approachability (Theorems 3.12, 3.18, and ??). And it applies to the weighted-majority games of resource allocation addressed in Section 4, where again a particular tie-breaking rule implies payoff approachability (Theorem 4.1), and further, for simple-majority games it implies existence of an equilibrium with zero probability of ties (Theorem 4.4).

Appendix A. Omitted Proofs from Section 2

Lemma 2.2. Assumption 2.1 implies that for each mixed strategy σ_j of a player j the set $\{ x_i \in X_i \mid \sigma_j(D(x_i)) = 0 \}$ is dense in X_i.

Proof. If the implication is not true, then there exists $\sigma_j \in \Sigma_j$ and an open set $V \subset X_i$ such that $\sigma_j(D(x_i)) > 0$ for all $x_i \in V$. By Assumption 2.1 and because V is open, we can find σ_i with $\sigma_i(V) > 0$ and $\sigma_i(D(x_j)) = 0$ for all $x_j \in X_j$. But then

$$0 = \int \sigma_i(D(x_j))\sigma_j(dx_j) = (\sigma_i \otimes \sigma_j)(D) = \int_V \sigma_j(D(x_i))\sigma_i(dx_i) + \int_{X_i \setminus V} \sigma_j(D(x_i))\sigma_i(dx_i) > 0,$$

a contradiction that establishes that the implication must be true. □

Lemma 2.4. If X_j is a finite dimensional manifold then Assumption 2.1 holds if, for each $x_i \in X_i$, $D(x_i)$ is a set of lower dimensionality in X_j.

Proof. For each x_j and each integer n, take U_n as a neighborhood of x_j that is diffeomorphic to the $1/n$-ball around the origin in $\mathbb{R}^m$, where m is the dimension of the manifold X_j. Let $\mu^n_{x_j}$ be the uniform distribution over this $1/n$-ball, so that $\mu^n_{x_j}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$. Let $f : U_n \to B_{1/n}(0)$ denote the diffeomorphism, and put $\eta^n_{x_j}(\cdot) = \mu^n_{x_j}(f(\cdot))$. Since $D(x_i)$ is a lower dimensional set in X_j, the Lebesgue measure $f(D(x_i))$ on $\mathbb{R}^m$ is zero. Hence $\eta^n_{x_j}(D(x_i)) = \mu^n_{x_j}(f(D(x_i)) = 0$ for every $x_i \in X_i$, so each x_i is a point a continuity against $\eta^n_{x_j}$. Since each $x_i \in X_i$ is also a point of continuity against convex combinations $\eta^n_{x_j}$'s, it is also a point of continuity against every σ^n_j in the convex hull Σ^n_j of the set of $\{ \eta^n_{x_j} \}_{x_j \in X_j}$. Let Σ^*_j be the union over n of Σ^n_j and note that $\Sigma^*_j \subset \{ \sigma_j \in \Sigma_j \mid \sigma_j(D(x_i)) = 0 \ \forall x_i \in X_i \}$. Observe that for each $\eta_j \in \Sigma_j$ with finite support there exists a sequence $\{ \sigma^k_j \} \subset \Sigma^*_j$ with $\sigma^k_j \to \eta_j$. It follows that $\Sigma^f_j \subset \mathrm{Cl}(\Sigma^*_j)$, where Σ^f_j is the set of mixed strategies with finite support and $\mathrm{Cl}(\cdot)$ denotes closure. As $\mathrm{Cl}(\Sigma^f_j) = \Sigma_j$, we have $\mathrm{Cl}(\Sigma^*_j) = \Sigma_j$, and the result follows. □

Theorem 2.6. If $\pi' = \pi^+$ or $\pi' = \pi^-$ then the value $v^*(\pi')$ exists. Moreover, in the game $G(\pi^+)$ player 1 has a maximin strategy, and in the game $G(\pi^-)$ player 2 has a minimax strategy.

Proof. We prove the theorem only for π^+ since the other case is similar. By Assumption 2.1 let $\tilde{\Sigma}_2$ be a dense set of strategies σ_2 such that $\sigma_2(D(x_1)) = 0$ for all $x_1 \in X_1$. We can assume without loss of generality that $\tilde{\Sigma}_2$ is a countable set: indeed, for each positive integer k, take a covering of Σ_2 by a finite number of balls of radius $1/k$, pick a point in each of these balls that belongs to $\tilde{\Sigma}_2$ and then take the countable union (over k) of these finite sets. Let $\tilde{\Sigma}_2^1 \subset \tilde{\Sigma}_2^2 \subset \cdots$ be an increasing sequence of subsets of $\tilde{\Sigma}_2$ such that each $\tilde{\Sigma}_2^n$ is a finite set and $\cup_n \tilde{\Sigma}_2^n = \tilde{\Sigma}_2$. For each n let Σ_2^n be the convex hull of $\tilde{\Sigma}_2^n$. Σ_2^n is a compact convex subset of Σ_2 for each n. Also, for each n and $\sigma_2^n \in \Sigma_2^n$, $\sigma_2^n(D(x_1)) = 0$ for all $x_1 \in X_1$; in particular, π^+ is continuous at each (x_1, σ_2^n) by Lemma 2.5.

Define a perturbed game G^n as follows. The strategy set of player 1 is Σ_1 and the strategy set of player 2 is Σ_2^n. The payoff function is the restriction of π^+ to $\Sigma_1 \times \Sigma_2^n$. The payoffs are clearly continuous and bilinear so the game G^n has a value, say v^n, and each player i has an equilibrium strategy σ_i^n that assures this value.

Take a convergent subsequence of equilibria (σ_1^n, σ_2^n) and associated values v^n converging to, say, (σ_1^*, σ_2^*) and v^* as $n \to \infty$. First observe that for each n, σ_2^n is a feasible strategy in $G(\pi^+)$ for player 2 that holds player 1's payoff down to v^n. Therefore, $v^n \geqslant \overline{v}(\pi^+)$ for all n, which implies that $v^* \geqslant \overline{v}(\pi^+)$. We now show that $\pi^+(\sigma_1^*, x_2) \geqslant v^*$ for all x_2, which implies that $v^* \leqslant \underline{v}(\pi^+)$. Indeed, otherwise there exists some x_2 such that $\pi^+(\sigma_1^*, x_2) < v^*$. In this case, we claim that we can assume without loss of generality that x_2 is a point of continuity against σ_1^*. To prove this claim, start with the given x_2 and first decompose σ_1^* into an average of two strategies σ_1^c and σ_1^d where x_2 is a point of continuity against σ_1^c and $\sigma_1^d(D(x_2)) = 1$. By Lemma 2.5, $\lim_{x_2^k \to x_2} \pi^+(\sigma_1^c, x_2^k) = \pi^+(\sigma_1^c, x_2)$. Moreover, $\pi^+(\sigma_1^d, x_2) = 1 \geqslant \pi^+(\sigma_1^d, x_2')$ for all x_2'. Therefore, for all x_2' sufficiently close to x_2, $\pi^+(\sigma_1^*, x_2') < v^*$. By Lemma 2.4 we can now choose a point x_2' close to x_2 such that it is a point of continuity against σ_1^* and also $\pi^+(\sigma_1^*, x_2') < v^*$. Thus, the claim is proved and we can assume that x_2 itself is a point of continuity against σ_1^*.

Because π^+ is continuous at $(\sigma_1^*, \delta_{x_2})$, pick $\varepsilon > 0$ and a neighborhood $U = U_1 \times U_2$ of $(\sigma_1^*, \delta_{x_2})$ such that for all $(\tau_1, \tau_2) \in U$, $\pi^+(\tau_1, \tau_2) < v^* - \varepsilon$. For all large n, $\sigma_1^n \in U_1$, and because $\tilde{\Sigma}_2$ is dense in Σ_2, there exists N and $\sigma_2 \in \tilde{\Sigma}_2^N$ that belongs to U_2 and thus $\pi^+(\sigma^n, \sigma_2) < v^* - \varepsilon$ for all large n. But, as the sequence $\tilde{\Sigma}_2^n$ is increasing, we have that for all $n \geqslant N$, σ_2 belongs to Σ_2^n and thus, $\pi^+(\sigma^n, \sigma_2) \geqslant v^n$, which is impossible as v^n converges to v^* and, as we just saw, $\pi^+(\sigma^n, \sigma_2) < v^* - \varepsilon$ for all large n. Thus $\pi^+(\sigma_1^*, x_2) \geqslant v^*$ for all x_2, as we wanted to show.

Thus we have shown that $\overline{v}(\pi^+) \leqslant v^* \leqslant \underline{v}(\pi^+) \leqslant \overline{v}(\pi^+)$, which shows that the game $G(\pi^+)$ has a value and that this value equals v^*. Moreover the fact that $\pi^+(\sigma_1^*, x_2) \geqslant v^*$ for all $x_2 \in X_2$ implies that σ_1^* is a maximin strategy. □

Theorem 2.9. If there exists a payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability then:

(1) $G(\tilde{\pi})$ has an equilibrium that yields the value $v^*(\tilde{\pi})$.
(2) For each $\varepsilon > 0$, each player j has a strategy σ_j^ε that is ε-optimal in $G(\tilde{\pi})$ and such that $\sigma_j^\varepsilon(D(x_i)) = 0$ for all $x_i \in X_i$.
(3) For each payoff function $\pi' \in \Pi$, the value $v^*(\pi')$ exists and is the same as $v^*(\tilde{\pi})$.

Proof. We divide the proof into intermediate steps. First we prove part (1).

Lemma A.1. *The game $G(\tilde{\pi})$ has an equilibrium and thus has a value $v^*(\tilde{\pi})$.*

Proof of Lemma. We show that $G(\tilde{\pi})$ satisfies better-reply security, and then existence follows from Reny [23, Corollary 5.2].[31] The players' payoff functions are reciprocally upper semi-continuous because the game is zero-sum, so it remains to show that the game is payoff secure (Reny [23, Definition, p. 1033]). For this, fix a mixed-strategy profile (σ_1, σ_2). For each player i, take a pure strategy x_i in the support of σ_i such that $\tilde{\pi}_i(x_i, \sigma_j) \geqslant \tilde{\pi}_i(\sigma_i, \sigma_j)$. By payoff approachability, for each $\varepsilon > 0$ there exists a point y_i close to x_i such that y_i is a point of continuity against σ_j and $\tilde{\pi}_i(y_i, \sigma_j) > \tilde{\pi}_i(x_i, \sigma_j) - \varepsilon/2$. Then, by Lemma 2.5, there exists a neighborhood U_j^ε of σ_j such that for each $\tau_j \in U_j^\varepsilon$, $\tilde{\pi}_i(y_i, \tau_j) > \tilde{\pi}_i(x_i, \sigma_j) - \varepsilon$, as required. □

Lemma A.2. *Consider a sequence of games $G(\tilde{\pi}^n)$, where $\tilde{\pi}^n$ is the restriction of $\tilde{\pi}$ to strategies in $\Sigma_1^n \times \Sigma_2^n \subset \Sigma$, and each sequence Σ_i^n converges to Σ_i in the Hausdorff topology on compact subsets of Σ_i. If each game $G(\tilde{\pi}^n)$ has an equilibrium σ^n and a value v^n, then v^n converges to $v^*(\tilde{\pi})$ and every limit point of σ^n is an equilibrium of $G(\tilde{\pi})$.*

Proof of Lemma. Take a convergent subsequence of equilibria σ^n and associated values v^n of $G(\tilde{\pi}^n)$ converging to say σ^* and v^*. We show that $v^* = v^*(\tilde{\pi})$ and that σ^* is an equilibrium of $G(\tilde{\pi})$, which proves the result. Fix a point x_1 for player 1 that is a point of continuity of σ_2^*. Fix $\varepsilon > 0$. Applying Lemma 2.5, there exists a neighborhood of $U_1 \times U_2$ of $(\delta_{x_1}, \sigma_2^*)$ such that $\tilde{\pi}(\sigma) \geqslant \tilde{\pi}(x_1, \sigma_2^*) - \varepsilon$ for all $\sigma \in U_1 \times U_2$. Since the strategy sets Σ_i^n converge to Σ, for all large n, there exists a strategy $\tau_1^n \in \Sigma_1^n \cap U_1$. Also, σ_2^n belongs to U_2 for large n. For such large n, as σ_2^n is an optimal strategy in $G(\tilde{\pi}^n)$, $v^n \geqslant \tilde{\pi}^n(\tau_1^n, \sigma_2^n)$, and thus $\tilde{\pi}(x_1, \sigma_2^*) - \varepsilon \leqslant \tilde{\pi}(\tau_1^n, \sigma_2^n) \leqslant v^n$, which implies that $\tilde{\pi}(x_1, \sigma_2^*) \leqslant \varepsilon + v^*$. Because ε is arbitrary, we conclude that $\tilde{\pi}(x_1, \sigma_2^*) \leqslant v^*$ for any x_1 that is a point of continuity against σ_2^*. Applying payoff approachability to player 1's payoffs shows that $\tilde{\pi}(x_1, \sigma_2^*) \leqslant v^*$ for *all* $x_1 \in X_1$ and thus that $v^* \geqslant v^*(\tilde{\pi})$. A similar argument with the roles of the players reversed shows that

[31] Alternatively, one can show that $G(\tilde{\pi})$ satisfies condition (C2) in Duggan [10], which also implies that $G(\tilde{\pi})$ is better-reply secure.

$v^* \leqslant v^*(\tilde{\pi})$ and thus $v^* = v^*(\tilde{\pi})$ as required. Thus σ_2^* is an optimal strategy for player 2 in $G(\tilde{\pi})$. Likewise, σ_1^* is optimal for player 1. Hence σ^* is a Nash equilibrium of $G(\tilde{\pi})$. □

Now we conclude the proof of the other parts of the theorem. We show that player 2 has a strategy as specified in part (2) of the theorem and that $\overline{v}(\pi') \leqslant v^*(\tilde{\pi})$ for all $\pi' \in \Pi$. A similar argument for player 1 completes the proof. As in the proof of Theorem 2.6, consider the perturbed games G^n where the strategies of player 2 are restricted to Σ_2^n. The strategy sets converge to the strategy sets in $G(\tilde{\pi})$ and thus Lemma A.2 applies. Take a convergent subsequence of equilibria (σ_1^n, σ_2^n) and associated values v^n converging to (σ_1^*, σ_2^*) and v^*. From Lemma A.2 we know that $v^* = v^*(\tilde{\pi})$.

For each $\varepsilon > 0$, choose n such that $v^n \leqslant v^*(\tilde{\pi}) + \varepsilon$. Since v^n is the value of G^n, $\pi(x_1, \sigma_2^n) \leqslant v^n \leqslant v^*(\tilde{\pi}) + \varepsilon$ for all x_1. By construction, $\sigma_2^n(D(x_1)) = 0$ for all x_1, and σ_2^n satisfies the properties specified in part (2) of the theorem. Also, observe that since $\sigma_2^n(D(x_i)) = 0$ for all x_i, no matter how payoffs are defined on D, the strategy σ_2^n holds player 1 down to $v^*(\tilde{\pi}) + \varepsilon$, i.e. $\overline{v}(\pi') \leqslant v^*(\tilde{\pi}) + \varepsilon$ for all π'. Since ε is arbitrary, $\overline{v}(\pi') \leqslant v^*(\tilde{\pi})$, as was to be shown. □

Remark A.3.

(1) Observe that the strategy profile σ^* constructed in the second part of the proof of Theorem 2.9 by invoking Lemma A.2 is actually an equilibrium of $G(\tilde{\pi})$. Thus part (1) can be viewed as a corollary to this result. Obtaining part (1) thus as a corollary of Lemma A.2 relies only on perturbation methods. We present the proof of part (1) separately, using better-reply security, to relate our results to previous literature on existence of equilibria in discontinuous games.

(2) If we had simply assumed that each X_i is a compact space then we could not have used the sequence $\tilde{\Sigma}_j^1 \subset \tilde{\Sigma}_j^2 \subset \cdots$ to construct a sequence of perturbed games. Rather, we would have needed a net $\{\, \tilde{\Sigma}_j^\alpha \,\}$ where the index α would be a collection of neighborhoods $\{\, U(x_j) \,\}_{x_j \in X_j}$ and $\tilde{\Sigma}_j^\alpha$ would be a finite subset of mixed strategies, one per open subset in a finite subcover of the collection. We would then use the corresponding net of perturbed games and the argument would proceed analogously.

Proposition 2.12. A payoff function $\tilde{\pi} \in \Pi$ satisfies payoff approachability if:

(1) For each i, x_i, $D(x_i)$ can be partitioned into finitely many Borel-measurable subsets $D^1(x_i), \ldots, D^n(x_i)$ such that for each $1 \leqslant l \leqslant n$:
   (a) $\tilde{\pi}_i(x_i, \cdot)$ is constant on $D^l(x_i)$.
   (b) For each closed $A^l \subseteq D^l(x_i)$, $\tilde{\pi}_i(y_i, \cdot)$ is constant on A^l for an open and dense set of y_i's in a neighborhood U of x_i.

(2) The condition in Definition 2.8 of payoff approachability holds for i, x_i and σ_j where the support of σ_j is finite and contained in $D(x_i)$.

Proof. Suppose that the conditions of the proposition are satisfied by a payoff function $\tilde{\pi}$. We show that $\tilde{\pi}$ satisfies payoff approachability. Fix (x_i, σ_j). We can decompose σ_j into an average of two strategies, σ_j^c and σ_j^d, where $\sigma_j^c(D(x_i)) = 0$ and $\sigma_j^d(D(x_i)) = 1$. For every sequence $x_i^n \to x_i$, we have that $\tilde{\pi}_i(x_i^n, \sigma_j^c) \to \tilde{\pi}_i(x_i, \sigma_j^c)$ as in Lemma 2.5. Thus the condition of Definition 2.8 is really about the property of $\tilde{\pi}_i(x_i, \sigma_j^d)$ and we can therefore assume without loss of generality that $\sigma_j = \sigma_j^d$, i.e. x_i is a point of discontinuity against every pure strategy in the support of σ_j.

Fix $\varepsilon > 0$. For each l choose a closed subset A^l of $D^l(x_i)$ such that $\sigma_j(A) \geqslant 1 - \varepsilon$, where $A = \cup_l A^l$. Let τ_j be the conditional distribution over A. It is sufficient to find a point y_i in the ε-ball around x_i such that y_i is a point of continuity against σ_j and $\tilde{\pi}_i(x_i, \tau_j) \leqslant \tilde{\pi}_i(y_i, \tau_j) + \varepsilon$. Indeed, using the fact that $\tilde{\pi}_i(x_i, x_j) \leqslant \tilde{\pi}_i(y_i, x_j) + 2$ for all x_j, this implies that $\tilde{\pi}_i(x_i, \sigma_j) - \tilde{\pi}_i(y_i, \sigma_j) \leqslant (1 - \varepsilon)\varepsilon + 2\varepsilon$, which proves the result.

Pick a point x_j^l in each A^l and define a mixed strategy $\tilde{\tau}_j$ as follows: $\tilde{\tau}_j(x_j^l) = \tau_j(A^l)$. The strategy $\tilde{\tau}_j$ has finite support by construction and also because $\tilde{\pi}_i(x_i, \cdot)$ is constant on each A^l by virtue of condition (1a), $\tilde{\pi}_i(x_i, \tau_j) = \tilde{\pi}_i(x_i, \tilde{\tau}_j)$. By condition (1b), we can choose a neighborhood U contained in the ε-ball around x_i such that $\tilde{\pi}_i(y_i, \cdot)$ is constant on each A^l for an open and dense set of y_i's in U. By Lemma 2.4, there exists $\tilde{y}_i$ in U such $\tilde{y}_i$ is a point of continuity against $\tilde{\tau}_j$ and $\tilde{\pi}_i(x_i, \tilde{\tau}_j) \leqslant \tilde{\pi}_i(\tilde{y}_i, \tilde{\tau}_j) + \varepsilon/2$. Because $\tilde{y}_i$ is a point of continuity against $\tilde{\tau}_j$ and using condition (1b) and Lemma 2.4 again, for σ_j, there exists a point y_i in U such that: (i) $\tilde{\pi}_i(y_i, \cdot)$ is constant on each A^l; (ii) y_i is a point of continuity against τ_j; (iii) $\tilde{\pi}_i(\tilde{y}_i, \tilde{\tau}_j) \leqslant \tilde{\pi}_i(y_i, \tilde{\tau}_j) + \varepsilon/2$. By (i), $\tilde{\pi}_i(y_i, \tau_j) = \tilde{\pi}_i(y_i, \tilde{\tau}_j)$. Assembling these inequalities and equalities,

$$\tilde{\pi}_i(x_i, \tau_j) = \tilde{\pi}_i(x_i, \tilde{\tau}_j) \leqslant \tilde{\pi}_i(\tilde{y}_i, \tilde{\tau}_j) + \varepsilon/2 \leqslant \tilde{\pi}_i(y_i, \tilde{\tau}_j) + \varepsilon = \tilde{\pi}_i(y_i, \tau_j) + \varepsilon,$$

which completes the proof. □

Appendix B. Proof of Theorem 4.4

We begin with a preliminary lemma about the payoff function $\tilde{\pi}$ that describes the tie-breaking rule $\mathcal{T}^S$, introduced in Definition 3.16 for simple-majority games. In this game, fix (x_i, σ_j) such that the support of σ_j is finite and contained in $D(x_i)$. Choose $\bar{\varepsilon}$ as in the proof of Theorem 3.12 and fix a neighborhood $V(x_i)$ also as there. The following lemma then proves payoff approachability for (x_i, σ_j) and, additionally, yields properties used to prove Theorem 4.4.

Lemma B.1. *There exists $k \in K^*(x_i)$ such that $\tilde{\pi}_i(x_i, \sigma_j) \leqslant \tilde{\pi}_i(y_i^k, \sigma_j)$. Moreover the inequality is strict if one of the following conditions holds:*

(1) *$\overline{K}(x_i)$ is nonempty and there is a positive probability of $(T2)$ or $(T3)$ being used.*
(2) *$K^*(x_i)$ has at least three coordinates and there is a positive probability of $(T2)$ or $(T3)$ being used.*
(3) *$K^*(x_i)$ has two coordinates and $(T2)$ or $(T3)$ is used in resolving a tie (x_i, x_j) for which $L^0(x_i, x_j) \neq \{ k \}$ for both k's in $K^*(x_i)$.*
(4) *$\overline{K}(x_i)$ is empty and $(T1)$ is invoked for some (x_i, x_j) because i satisfies the conditions for the rule and either: $|L^0(x)| \geqslant 3$ and $\tilde{\pi}_i(y_i^{k'}, x_j) = 0$ for some $k' \in K^*(x_i)$; or $u_{k''}(x_i) \neq u_{k''}(x_j)$ for some $k'' \in K^*(x_i)$.*

Proof. The proof becomes transparent once we compare the payoffs to x_i and y_i^k against x_j for each k and x_j, which we now do.

If $(T1)$ is invoked and $\tilde{\pi}_i(y_i^k, x_j)$ is 0 (resp. -1) for some $k \in L^*(x)$, then $\tilde{\pi}_i(y_i^{k'}, x_j)$ is 0 (resp. -1) for all k' in $L_i^*(x)$, because of simple-majority scoring, and $\tilde{\pi}_i(y_i^{k'}, x_j)$ is 1 (resp. non-negative) for $k' \in K^*(x_i) \setminus L_i^*(x)$. Thus in this case $\tilde{\pi}_i(x_i, x_j) \leqslant \tilde{\pi}_i(y_i^k, x_j)$ for all $k \in K^*(x_i)$, with strict inequality if $\overline{K}(x_i)$ is empty and either: (i) $|L^0(x)| \geqslant 3$ and $\tilde{\pi}_i(y_i^{k'}, x_j) = 0$ for some $k' \in K^*(x_i)$; or (ii) $u_k(x_i) \neq u_k(x_j)$.

If $(T1)$ is invoked because $\tilde{\pi}_j(y_j^k, x_i)$ is 0, then $\tilde{\pi}_i(x_i, x_j)$ is zero if $|L^0(x)| = 2$ and $+1$ if $|L^0(x)| \geqslant 3$. By Assumption 3.14, $\tilde{\pi}_i(y_i^k, x_j)$ is nonnegative in the former case and is $+1$ in the latter. Likewise, if $(T1)$ is invoked because $\tilde{\pi}_j(y_j^k, x_i)$ is -1, then $\tilde{\pi}_i(y_i^k, x_j) = +1$ by Assumption 3.14. In short, $\tilde{\pi}_i(y_i^k, x_j) \geqslant \tilde{\pi}_i(x_i, x_j)$ for all k. Thus, y_i^k does at least as well as x_i against every x_j for which $(T1)$ is applied.

There remains to consider x_j's for which $(T2)$ or $(T3)$ is invoked.

Suppose $L_i^*(x)$ is empty for each i. If $|L^j(x)| = K/2$ then $\tilde{\pi}_i(x_i, x_j) = -1/2$ from $(T2)$. For any $k \in K^*(x_i)$, because $k \notin L^0(x)$, $u_{k'}(y_i^k) > u_{k'}(x_j)$ for all $k' \in L^0(x)$, so $|L^i(y_i^k, x_j)| = K/2$ as well, and $\tilde{\pi}_i(y_i^k, x_j) = 0$. Likewise, if $|L^i(x)| = K/2$, then $\tilde{\pi}_i(x_i, x_j) = 1/2$ from $(T2)$, and because $k \notin L^0(x)$, $\tilde{\pi}_i(y_i^k, x_j) = 1$. Summing up, $\tilde{\pi}_i(y_i^k, x_j) - \tilde{\pi}_i(x_i, x_j) = 1/2$ if either $|L^i(x)|$ or $|L^j(x)|$ equals $K/2$. This difference is equal to $+1$ otherwise (i.e. if neither of the candidates has half the votes outside of $L^0(x)$). Thus all y_i^k's do strictly better against all these x_j's.

Suppose $L^0(x)$ contains just one voter, say k. If $k \notin K^*(x_i)$, then the payoff difference is as in the previous paragraph. If $k \in K^*(x_i)$, then $\overline{K}(x_i)$ is empty (by point (1) of Assumption 3.14) and $\tilde{\pi}_i(y_i^k, x_j) - \tilde{\pi}_i(x_i, x_j) = -1 + 1/2 = -1/2$ (resp. $0 - 1/2 = -1/2$) if $|L^j(x)| = K/2$ (resp. $|L^i(x)| = K/2$). This difference is equal to -1 if neither of the candidates has half of the voters outside of $L^0(x)$. But observe that for every $k' \neq k$ in $K^*(x_i)$, $\tilde{\pi}_i(y_i^{k'}, x_j) - \tilde{\pi}_i(x_i, x_j) = \tilde{\pi}_i(x_i, x_j) - \tilde{\pi}_i(y_i^k, x_j)$, as $u_k(y_i^{k'}) > u_k(x_j) > u_k(y_i^k)$. Thus each $y_i^{k'}$ does strictly better against all these x_j's.

Finally suppose $L^0(x)$ contains at least two voters, either $L_i^*(x)$ or $L_j^*(x)$ is nonempty but each player for whom it is nonempty that he can achieve $+1$ rather than 0 or -1 specified there. Then if $L_i^*(x)$ is nonempty $\tilde{\pi}_i(y_i^k, x_j) = 1$ for some $k \in L_i^*(x)$ (otherwise $(T1)$ would apply) and it holds for all k while $\tilde{\pi}_i(x_i, x_j) = 0$; on the other hand if $L_i^*(x)$ is empty, then trivially each y_i^k achieves $+1$.

We now complete the proof of the lemma as follows. Obviously if $\overline{K}(x_i)$ is nonempty, then y_i^k does at least as well as x_i against each x_j in the support of σ_j and strictly better against all x_j's for which $(T1)$ is not invoked, proving the first statement and points (1-3) of the second, with point (4) being vacuously true. Assume from now on that $\overline{K}(x_i)$ is empty.

Each y_i^k does as well against all x_j to which $(T1)$ applies and strictly better against those x_j's for which the condition of point (4) of the lemma holds. If $(T2)$ or $(T3)$ is not used with positive probability then the first statement of the lemma holds as does point (4), while points (1-3) are vacuous.

Suppose $(T2)$ or $(T3)$ is invoked with positive probability. If there is one k for which no tie is just on this voter's utility, then y_i^k does strictly better than x_i as the calculations above show. Thus, the inequality holds, regardless of the conditions of points (2)-(4), if there is such a k. Suppose then that for each $k \in K^*(x_i)$ there is an x_j that ties with x_i just on k. It is clear that at least one of the y_i^k's would do as well as x_i against σ_j. Moreover, if there are at least three coordinates in $K^*(x_i)$, one of them would do strictly better, proving point (2). Also, if there are only two such k's then one of them would do strictly better than x_i against σ_j unless each tie involves exactly one of these k's, which proves point (3). Observe that when there are two such k's, and x_i is not inferior to some y_i^k against σ_j, then x_i and each y_i^k give the same payoff against the conditional distribution over the x_j's for which $(T2)$ or $(T3)$ is used.

Coming to ties involving $(T1)$ it is clear now that if there is a tie with an x_j where the rule is invoked because of i, then for x_i to do at least as well as all y_i^k, we must have $K^*(x_i) \subset L^0(x)$ and $\tilde{\pi}_i(y_i^k, x_j) = -1$ for each $k \in K^*(x_i)$ if $|L^0(x)| > 2$. If this is violated for some x_i and if x_i is already not dominated by some y_i^k against the conditional over x_j's where $(T1)$ is not used, then $K^*(x)$ has two coordinates and as we argued at the end of the last paragraph each k would do equally well against those not involving $(T1)$, with the result that it would do strictly better against σ_j, proving point (4). □

We now recall and prove Theorem 4.4 for simple-majority Colonel-Blotto games.

Theorem 4.4. Let σ^* be an equilibrium that is invariant under all the symmetries of the game. If $R_1/R_2 < r^*$ then $(\sigma_1^* \otimes \sigma_2^*)(D) = 0$, that is, at the equilibrium σ^* the tie-breaking rule $\mathcal{T}^S$ has zero probability of being invoked.

We set up some notation and prove a number of preliminary claims before proving the theorem. Suppose x_i is a strategy in X_i such that $\sigma_j^*(D(x_i)) > 0$. We can decompose σ_j^* into σ_j^{c,x_i} and σ_j^{d,x_i}, where the former puts zero probability on $X_j \setminus D(x_i)$ and the latter puts probability one on it. Let $\mathcal{L}(x_i)$ be the set of quadruples $L = (L^0, L^i, L^j, Tn)$ such that there is a positive probability under σ_j^* of the set $D^L(x_i)$ consisting of x_j's such that $(L^0, L^i, L^j) = (L^0(x_i, x_j), L^i(x_i, x_j), L^j(x_i, x_j))$ and provision (Tn) of rule $\mathcal{T}^S$ is used, where $n \in \{1, 2, 3\}$. For simplicity, from here on we suppress Tn in the notation. For each L choose a point $x_j(L) \in D^L(x_i)$ and consider the conditional distribution $\tilde{\sigma}_j^{x_i}$ over the $x_j(L)$'s given by $\tilde{\sigma}_j^{x_i}(x_j(L)) = \left(\sum_{L'} \sigma_j^*(D^{L'}(x_i)\right)^{-1} \sigma_j^*(D^L(x_i))$. Choose a neighborhood $V(x_i)$ such that for each $y_i \in V(x_i)$ and L, $y_{i,k} > x_{j,k}(L)$ if $k \in L^i$, and $y_{i,k} < x_{j,k}(L)$ if $k \in L^j$.

Claim B.2. $\tilde{\pi}_i(x_i, \sigma_j^*) = \sigma_j^*(X_j \setminus D(x_i))\tilde{\pi}_i(x_i, \sigma_j^{c,x_i}) + \sigma_j^*(D(x_i))\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i})$.

Proof. As the payoff $\tilde{\pi}_i(x_i, \cdot)$ is constant on each $D^L(x_i)$, $\tilde{\pi}_i(x_i, \sigma_j^{d,x_i}) = \tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i})$ and the result follows. □

Claim B.3. *If x_i is a best reply to σ_j^*, then $\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i}) \geqslant \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i})$ for all $k \in K^*(x_i)$.*

Proof. Assume to the contrary that $\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i}) < \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i})$ for some $k \in K(x_i)$. For each $\varepsilon > 0$ let $W^\varepsilon(x_i)$ be the set of y_i such that $|y_{i,k} - x_{i,k}| < \varepsilon$. For each L, let $D^{\varepsilon,L}(x_i)$ be the set of x_j in $D^L(x_j)$ such that $|x_{i,k} - x_{j,k}| > \varepsilon$ for $k \notin L^0$ and let $D^\varepsilon(x_i)$ be the union of the $D^{\varepsilon,L}(x_i)$'s. Choose ε small enough such that each $x_j(L)$ belongs to $D^\varepsilon(x_i)$. Define $\tilde{\sigma}_j^{\varepsilon,x_i}$ to be the distribution over $x_j(L)$ that assigns probability $\sigma_j^{d,x_i}(D^{\varepsilon,L}(x_i))/\sum_{L'} \sigma_j^{d,x_i}(D^{\varepsilon,L'}(x_i)$ to $x_j(L)$. By construction $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \cdot)$ is constant on the set $D^{\varepsilon,L}(x_i)$ for each L and $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), x_j) \in [-1, 1]$ for all x_j. Hence,

$$\tilde{\pi}_i(y_i(W^\varepsilon(x_i), \sigma_j^{d,x_i}) \in (\sigma_j^{d,x_i}(D^\varepsilon(x_i)))\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \tilde{\sigma}_j^{\varepsilon,x_i}) \pm \sigma_j^{d,x_i}(D(x_i) \setminus D^\varepsilon(x_i)) \, .$$

Obviously $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), x_j(L)) = \tilde{\pi}_i(y_i^k, x_j(L))$ for all $x_j(L)$. Moreover, $\tilde{\sigma}_j^{\varepsilon,x_i}$ converges to $\tilde{\sigma}_j^{x_i}$ and $D^\varepsilon(x_i)$ converges to $D(x_i)$. Therefore, $\lim_{\varepsilon\downarrow 0} \tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \sigma_j^{d,x_i}) = \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i}) > \tilde{\pi}_i(x_i, \sigma_j^{d,x_i})$. Since $\lim_{\varepsilon\downarrow 0} \tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \sigma_j^{c,x_i}) = \tilde{\pi}_i(x_i, \sigma_j^{c,x_i})$, we then have that $\tilde{\pi}_i(x_i, \sigma_j^*) < \lim_{\varepsilon\downarrow 0} \tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \sigma_j^*)$ and σ_j^* is not a best reply to σ_j^*, a contradiction. □

The next three claims argue directly about points $(x_i, x_j) \in D$.

Claim B.4. *If x_i is a vertex, then there exists x'_j obtained by permuting the coordinates of x_j such that $(T1)$ does not apply to (x_i, x'_j).*

Proof. Let x_i be a strategy that assigns R_i to a battle, say $k = 1$. Observe first that for $(T1)$ to be used in deciding a tie between x_i and x_j's, this battle must belong to $L^0(x)$. If $R_1 = R_2$, this means that $x_i = x_j$ and $(T3)$ is operative. If $R_1 > R_2$, then $i = 2$ and $\tilde{\pi}_i(y_i^1, x_j) = -1$. Since $R_1 < r^* R_2$, there exists some $k' \neq 1$ such that $0 < x_{j,k'} < R_2$. There exists some x'_j that swaps these two coordinates and now $(T3)$ applies to (x_i, x'_j). □

Claim B.5. *Suppose x_i is not a vertex, and $(T1)$ applies to $(x_i, x_j) \in D$. If $\tilde{\pi}_i(y_i^k, x_j)$ is either 0 or -1 for some $k \in L^*(x)$, then either: (i) there exists $k' \in K^*(x_i)$ such that $x_{i,k'} \neq x'_{j,k'}$ for some x'_j obtained from permuting the coordinates of x_j; or (ii) $|L^0(x)| \geqslant 3$ and $\tilde{\pi}_i(y_i^k, x_j) \geqslant 0$ for some $k \in L_i^*(x)$.*

Proof. If $R_1 = R_2$, conclusion (i) is valid, since otherwise $x_i = x_j$ and $(T3)$ would apply. If $R_1 > R_2$ and $i = 1$, then conclusion (i) is obvious.

Assume now that $i = 2$, $R_1 > R_2$ and conclusion (i) of the claim is violated. Then $x_{i,k} = x_{j,k}$ for each positive coordinate of x_i. If $\tilde{\pi}_i(x_i, x_j) = 0$ for some k, then K is even, $|L^0(x)| = 2$, and $|L^j(x)| = K/2 - 1$, while if $\tilde{\pi}_i(x_i, x_j) = -1$, then either $|L^j(x)| = \lfloor K/2 \rfloor$ (K can be odd or even) or $|L^0(x)| \geqslant 3$ K is even and $|L^j(x)| = K/2 - 1$. If $|L^0(x)| = 2$, then $|L^j(x)| = K - |L^0(x)| = K - 2 > \lfloor K/2 \rfloor - 1$. Thus, when $|L^j(x)| = K/2 - 1$, $|L^0(x)| \geqslant 3$.

If $|L^j(x)| = \lfloor K/2 \rfloor$, then $|L^0(x)| = \lceil K/2 \rceil$. Therefore, there exists k' such that $x_{i,k'} \geqslant R_2/\lceil K/2 \rceil$. Moreover, since $|L^j(x)| = \lfloor K/2 \rfloor$, and $R_1 < r^* R_2$, there exists a coordinate k'' such that $x_{i,k''} = 0 < x_{j,k''} < R_1 - R_2 < R_2/\lceil K/2 \rceil$. There exists x'_j that swaps these two coordinates and $(x_i, x'_j) \in D$. Now there is a coordinate, namely k', for which $x_{i,k'} > x'_{j,k'}$, a contradiction. So (i) must hold.

If $|L^j(x)| = K/2 - 1$ then, as we saw above, $|L^0(x)| \geqslant 3$. Therefore, $\tilde{\pi}_i(y_i^k, x_j) = 0$ for each $k \in L_i^*(x)$, which proves (ii). □

Claim B.6. *Suppose $(x_i, x_j) \in D$, both x_i and x_j have two positive coordinates, $L^*(x_i)$ is nonempty, and $(T2)$ or $(T3)$ applies. There exists another x'_j obtained by a permutation of coordinates from x_j where $(T2)$ or $(T3)$ applies as well but where (x_i, x'_j) are either tied in two or more coordinates or in a zero coordinate.*

Proof. Suppose x_i and x_j are tied in just coordinate, say $k = 1$, and that this coordinate is positive for both players. Then $K = 3$ and i wins, say, $k = 2$ and j wins $k = 3$. Derive x'_j from x_j by permuting coordinates 2 and 3. x'_j ties with x_i in coordinates 1 and 3. □

Proof of Theorem 4.4. Fix $x_1 \in D$ such that $\sigma_j^*(D(x_i)) > 0$. We show that x_i is not a best reply to σ_j^*, which proves the result.

Fix x_j in $D(x_i)$. Let $L = (L^0(x), L^i(x), L^j(x))$. Observe that if x_j' is obtained by permuting coordinates of x_j, then there exists $x_j'(L')$ in the support of $\tilde{\sigma}_j^{x_i}$ where $L' = (L^0(x_i, x_j'), L^i(x_i, x_j'), L^j(x_i, x_j'))$. Using this fact, the proof of the theorem follows quite easily. If x_i is a vertex, by Claim A.4, point (1) of Lemma B.1 holds for $\tilde{\sigma}_j^{x_i}$, and by Claim B.3, x_i is not a best reply to σ_j^*.

The other cases work similarly. If x_i is not a vertex, but $(T1)$ applies to (x_i, x_j), then combining Claim A.5, point (4) of Lemma B.1 and Claim B.3 proves the result.

If $(T2)$ or $(T3)$ applies to (x_i, x_j), then by point (2) of Lemma B.1, x_i has only two non-zero coordinates. Claim A.6, point (3) of Lemma B.1, and Claim B.3 finish the proof. □

References

[1] Banks, J. and J. Duggan (2008): A Dynamic Model of Democratic Elections in Multidimensional Policy Spaces, Quarterly Journal of Political Science, 3, 269-299.

[2] Banks, J., J. Duggan, and M. Le Breton (2002), Bounds for Mixed Strategy Equilibria and the Spatial Model of Elections, Journal of Economic Theory, 103, 88-105.

[3] Blackwell, D. (1956): An Analog of the Minmax Theorem for Vector Payoffs, Pacific Journal of Mathematics, 6, 1-8.

[4] Billingsley, P. (1999): Convergence of Probability Measures, Second Edition. Wiley, New York.

[5] Borel, E. (1953): Theory of Play and Integral Equations with Skew Symmetric Kernels (English translation of 'La Theorie du Jeu et les Equations Integrales a Noyan Symetrique,' in Comptes Rendus de d'Academie des Sciences, 173, 1304-1308), Econometrica, 21, 97-100.

[6] Borel, E., and J. Ville (1938): Application de la Theorie des Probabilites aux Jeux de Hasard. Paris: Gauthier-Villars. Reprinted in E. Borel and A. Cheron (1991): Theorie Mathematique du Bridge á la Portee de Tous, Paris: Editions Jacques Gabay.

[7] Dasgupta, P., and E. Maskin (1986): The Existence of Equilibrium in Discontinuous Economic Games 1: Theory, Review of Economic Studies, 53, 1-27.

[8] Diermeier, D., and R. Myerson (1999): Bicameralism and Its Consequences for the Internal Organization of Legislatures, American Economic Review, 89, 1182-1196.

[9] Downs, A. (1957): An Economic Theory of Democracy. Harper and Row, New York.

[10] Duggan, J. (2007): Equilibrium Existence for Zero-Sum Games in Spatial Models of Elections, Games and Economic Behavior, 60, 52-74.

[11] Duggan, J. (2011): Private communication.

[12] Duggan, J., and M. Jackson (2005): Mixed Strategy Equilibrium and Deep Covering in Multidimensional Electoral Competition, University of Rochester, mimeo.
URL: http://www.stanford.edu/~jacksonm/mseuc.pdf

[13] Gross, O., and R. Wagner (1950): A Continuous Colonel Blotto Game, Research Memorandum RM-408. Santa Monica, CA: Rand Corporation.

[14] Hart, S. (2008): Discrete Colonel Blotto and General Lotto Games, International Journal of Game Theory, 36, 441-460.

[15] Jackson, M., and J. Swinkels (2005): Existence of Equilibrium in Single and Double Private Value Auctions, Econometrica, 73, 93-139.

[16] Kvasov, D. (2007): Contests with Limited Resources, Journal of Economic Theory, 136, 738-748.

[17] Laslier, J.-F. (2002): How Two-Party Competition Treats Minorities, Review of Economic Design, 7, 297-307.

[18] Laslier, J.-F. (2005): Party Objectives in the 'Divide a Dollar' Electoral Competition, Social Choice and Strategic Decisions, Essays in Honor of Jeff Banks, D. Austen-Smith and J. Duggan (eds.), 113-130. New York: Springer.

[19] Laslier, J.-F. and N. Picard (2002): Distributive Politics and Electoral Competition, Journal of Economic Theory, 103, 106-130.

[20] Mertens, J.-F. (1986): The Minmax Theorem for U.S.C.-L.S.C. Payoff Functions, International Journal of Game Theory, 15, 237-250.

[21] Parthasarathy, T. (1970): On Games Over the Unit Square, SIAM Journal on Applied Mathematics, 19, 473-476.

[22] Plott, C. (1967): A Notion of Equilibrium and its Possibility under Majority Rule, American Economic Review, 57, 787-806.

[23] Reny, P. (1999): On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games, Econometrica, 67, 1029-1056.

[24] Roberson, B. (2006): The Colonel Blotto Game, Economic Theory, 29, 1-24.

[25] Roberson, B., and D. Kvasov (2012): The Non-Constant-Sum Colonel Blotto Game, Economic Theory, 51, 397-434.

[26] Simon, L., and W. Zame (1990): Discontinuous Games and Endogenous Sharing Rules, Econometrica, 58, 861-872.

[27] Sion, M., and P. Wolfe (1957): On a Game Without a Value, Contributions to the Theory of Games, III, 299-306. Princeton: Annals of Mathematical Studies No. 39.

[28] Thomas, C. (2012): N-dimensional Blotto Game with Asymmetric Battlefield Values, The University of Texas at Austin, mimeo.

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