The Generalized Random Priority Mechanism with Budgets*

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Abstract

This paper studies allocation problems with and without monetary transfers, such as multi-unit auctions, school choice, and course assignment. For this class of problems, we introduce a generalized random priority mechanism with budgets (GRP). This mechanism is always ex post incentive compatible and feasible. Moreover, as the market grows large, this mechanism can approximate any incentive compatible mechanism in the corresponding continuum economy. In particular, GRP can be used to approximate efficient and envy-free allocations, while preserving incentive compatibility and feasibility.

1 Introduction

In this paper we study allocation problems with indivisible goods, such as multi-unit auctions, school choice, course assignment, etc. We introduce a generalized random priority mechanism with budgets (GRP), aimed at achieving the following properties: ex post incentive compatibility, feasibility, and, as the market grows large, approximate efficiency and envy-freeness. The mechanism is applicable to environments with and without monetary transfers and can accommodate interdependent valuations (as long as agents' signals are independent conditionally on the true state of the world).

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The goal of finding a mechanism that achieves these properties is motivated by two observations. On one hand, several negative results are known for finite economies. For instance, full efficiency is generally impossible to achieve with incentive-compatible mechanisms when agents have interdependent values and receive multidimensional signals (Maskin, 1992; Jehiel and Moldovanu, 2001; Jehiel et al., 2006; Hashimoto, 2008; Che et al., 2012). Moreover, when monetary transfers cannot be used, efficiency and fairness cannot coexist with incentive compatibility even when values are private (Zhou, 1990; Bogomolnaia and Moulin, 2001).

On the other hand, in infinite (continuum) economies, where one agent's actions do not affect the allocations of others, the set of outcomes that are implementable by an incentive compatible and feasible mechanism is much larger. In particular, in such settings, efficiency is much easier to achieve. This contrast gives rise to a natural question: In markets that are large (but finite), is it possible to approximate the corresponding infinite-market mechanism (e.g., by slightly relaxing full efficiency) in such a way that incentive compatibility and feasibility are preserved? Our results provide a constructive positive answer to this question: GRP is a way to achieve it.

1.1 Main Results

Our main finding is as follows:

> *Under certain conditions, any feasible (ex post) incentive-compatible large-market mechanism can be asymptotically approximated by a finite-market mechanism that is feasible and ex post incentive compatible.*

This approximation should be interpreted as a convergence in probability: With a probability almost equal to one, the payoffs and revenues in finite markets are (uniformly) very close to those in infinite markets.

We employ the GRP mechanism with budgets in the approximation. The random priority, as its name suggests, randomly assigns strict priorities to agents. From the top of the ordering, agents sequentially and dictatorially choose their optimal consumption bundles from the remaining objects. This mechanism is clearly feasible and strategy-proof (dominant strategy incentive compatible) in the case of private values (Abdulkadiroğlu and Sönmez, 1998). Importantly, these properties are inherited by GRP, which by construction satisfies the requirements of feasibility and ex post incentive compatibility.

Like the random priority, GRP randomly prioritizes agents based on which consumption bundles are sequentially assigned to them. Before the sequential assignment starts, the mechanism gathers reports from agents and then assigns to each agent a *budget set*. This set consists of the options *potentially* available to the agent, and it may depend on any report except for the agent's own report. The mechanism then begins assignment according to the following process. From the top of the priority, each agent lets the mechanism solve her decision problem. The mechanism chooses an optimal option from the budget set, but an option is available *only when* it is always feasible regardless of the choice of the agent's report. This feasibility constraint keeps GRP always feasible. It is also ex post incentive compatible because both the budget set and feasibility constraint are independent of the corresponding agent's own report, as is further described in Section 4.

We emphasize the feasibility of the approximating mechanism. Feasibility is often considered to be obligatory—and this paper essentially sticks to this view—but sometimes desirable mechanisms are found in the class of infeasible ones (Cordoba and Hummond, 1998; Kovalenkov, 2002; Budish, 2011). In finite markets, we always use GRP to ensure that infeasibility never occurs.

1.2 Application 1: Combinatorial Auctions

We apply our approximation method to two interesting cases. Our first application is combinatorial auctions. We establish the following in Section 6.2:

> *A Walrasian equilibrium exists in continuous markets under a certain non-atomic condition. There exists a finite-market mechanism that approximates equilibrium outcomes in large finite markets. The approximating mechanism is feasible, ex post incentive compatible, asymptotically surplus maximizing, and asymptotically envy-free.*

This result asserts that *full efficiency is nearly achievable even with ex post incentive compatibility*. This finding contrasts with the impossibility results of Maskin (1992), Jehiel and Moldovanu (2001), Jehiel et al. (2006), and Hashimoto (2008). All these studies assume that agents have interdependent values and multidimensional signals. The first two studies show that it is almost impossible for Bayesian incentive compatible mechanisms to achieve full efficiency, while the latter two even claim that any ex post incentive-compatible mechanism is almost always constant if goods cannot be wasted; in other words, such a mechanism simply follows a predetermined assignment plan (i.e., it ignores reports from agents) and thus is highly inefficient. Bikhchandani (2006) has already pointed out that this generic constancy result breaks down when goods are private. However, the degree to which ex post incentive-compatible mechanisms could approach the efficiency frontier in this case remained unanswered.

When valuations are private and satisfy gross substitutability, mechanisms with desirable properties do exist (Ausubel, 2006; İnal, 2011). However, without this restriction, they become harder to construct. For example, Vickery outcomes may generate very low revenues for the sellers and thus will fall outside of the core (Ausubel and Milgrom, 2002). Partly motivated by this problem, Day and Milgrom (2008) introduce a new scheme, which they term *core-selecting package auctions*. These auctions are similar to the asymptotically Walrasian mechanism in that both types of mechanisms always use, or at least aim to use, outcomes in the core. Like Day and Milgrom (2008), we do not impose the gross-substitute condition on preferences. Unlike Day and Milgrom (2008), however, we focus on incentive compatible mechanisms, whereas they focus on fully efficient mechanisms that may not be incentive compatible.

Notably, we do not impose such assumptions on individual preferences as concavity, gross substitution, or any other restriction of that nature. We instead need a certain distributional nonatomic condition, but we can allow any sort of complementarity in individual utility functions. In this sense, this paper studies a more general preference domain compared to the literature on finite-economy Walrasian equilibria, which rely on the gross substitute condition (Gul and Stacchetti, 1999; Ausubel, 2006; İnal, 2011).

A lattice-theoretical single-crossing condition, as appears in Bikhchandani (2006), is also unnessesary at the ex ante stage when utility functions depend on signals. However, we do need a weak nonconstancy condition with respect to signals, which is almost identical to that described by Jehiel et al. (2006) when agents have only two choices.

1.3 Application 2: Multi-Unit Assignment without Money

Our method also applies to allocation problems in which agents potentially have multi-unit demands and monetary transfers are never used, such as in the examples of school choice and course assignment. In Section 6.3, we approximate an extension of the *Hylland-Zeckhauser (HZ) equilibrium*, the equilibrium concept proposed by Hylland and Zeckhauser (1979). Considering finitely many agents with single-unit demands, the authors define this equilibrium concept in hypothetical markets where agents purchase lotteries of goods using pseudomoney. We extend the HZ equilibrium to the case with a continuum of agents who may have multi-unit demands, establishing the following result:

> *A generalized Hylland-Zeckhauser equilibrium exists in continuous markets. There exists a mechanism that approximately achieves equilibrium outcomes in large finite markets. The approximating mechanism is feasible, ex post incentive compatible, asymptotically efficient, and asymptotically envy-free.*

This result may come as a surprise, especially when agents have multi-unit demands. First of all, the Hylland-Zeckhauser equilibrum does not immediately extend to multi-unit demands. As explained above, agents purchase individual-level lotteries in the pseudomarket, but these lotteries must be consistently implemented by an economy-level lottery. For example, two identical lotteries that give each of two agents a 50% chance of winning a unique item cannot be independently implemented; instead, they must be perfectly negatively correlated. As long as agents demand at most one unit, the existence of such an economy-level lottery is ensured by the Birkhoff–von Neumann theorem (Hylland and Zeckhauser, 1979;

Bogomolnaia and Moulin, 2001). Budish et al. (2012) successfully extend HZ equilibria to multi-unit demands by restricting the class of possible preferences. This class of preferences, however, does preclude certain complementarities; e.g., $v(1,0) = v(0,1) = 1$ and $v(1,1) = 3$ is not allowed. In this paper, we do not impose any such restrictions on preferences.

In finite markets, a number of papers have studied multi-unit assignment problems. Among strategy-proof mechanisms, the serial dictatorship emerges as the unique deterministic efficient mechanism with additional technical properties (Pápai, 2001; Ehlers and Klaus, 2003), but the mechanism is often considered to be unfair. Two studies propose auction- or market-like mechanisms with pseudomoney (Sönmez and Ünver, 2010; Budish, 2011). Sönmez and Ünver (2010) propose an auction-like mechanism based on the agent-proposing deferred acceptance mechanism but their mechanism may not be efficient. Meanwhile, Budish (2011) proposes a market mechanism that possesses desirable properties in efficiency and fairness, but generally induces infeasible allocations. Further, Kojima (2009) extends the probabilistic serial mechanism (Bogomolnaia and Moulin, 2001) to the case with multi-unit demands and additively separable utility functions, but both the extension and the original are efficient only in the *ordinal* sense rather than in the much stronger *cardinal* sense.

Even when agents have single-unit demands and private values, it is not immediately clear how closely "satisfactory" strategy-proof mechanisms can approach full efficiency. As shown by Zhou (1990), exact efficiency is never achieved by a symmetric strategy-proof mechanism. In contrast, *ordinal* efficiency is asymptotically achieved by strategy-proof mechanisms, as demonstrated by Che and Kojima (2010) and Liu and Pycia (2012). However, ordinal efficiency is weaker than our definition of efficiency (i.e., cardinal efficiency).

1.4 Related Literature

The recent study by Azevedo and Budish (2012) also considers incentives and approximation in finite and infinite markets, but their approach contrasts with that described in the present paper. These authors consider a finite-market mechanism with a sequence of Bayesian equilibria indexed by market size. Their main theorem asserts that a sequence of equilibrium outcomes can be approximated by using the truth-telling outcomes of another mechanism that is asymptotically strategy-proof. Our approach differs from theirs in two ways. First, we consider exact (ex post) incentive-compatibility rather than asymptotic incentive-compatibility, even in finite markets. Second, we start from the existence of an incentive compatible large-market mechanism, whereas Azevedo and Budish (2012) start from a sequence of Bayesian equilibria in finite markets.

Two studies propose (almost) strategy-proof mechanisms that approximate Walrasian equilibria in pure exchange economies with divisible goods (Córdoba and Hammond, 1998; Kovalenkov, 2002). However, Córdoba and Hammond's mechanism is only asymptotically strategy-proof, while Kovalenkov's is only asymptotically feasible. In contrast, our mechanism is always both feasible and strategy-proof regardless of market size.

Segal (2003) also puts forward the idea of the asymptotic approximation in the context of optimal auctions. Assuming that the distribution of i.i.d. signals is unknown, he demonstrates that the revenues from the optimal auction asymptotically achieve the maximum revenues that are achievable when the distribution is known. He focuses on private values, whereas we allow interdependent values in this paper.

Approximation problems with Bayesian incentive compatibility have been considered by the series of papers by Gul and Postlewaite (1992) and McLean and Postlewaite (2002, 2003a,b, 2004, 2005). These papers study the approximation of full efficiency or incomplete-information core when market size is large or agents are informationally small. They assume finitely many states and types, which preclude subtle difficulties in approximation, whereas the present paper considers continuously many types.

1.5 Organization of the Paper

The remainder of this paper is organized as follows. Section 2 provides a simple auction example that illustrates the idea of the approximation theorem. We present our model in Section 3. GRP and its properties are described in Section 4. Section 5 presents the approximation theorems. In Section 6, we apply these approximation theorems to combinatorial auctions (Section 6.2) and multi-unit assignment without monetary transfer (Section 6.3). Section 7 discusses related issues and concludes.

2 Example

We start with an example that illustrates the main idea of the paper. There are m identical indivisible objects and n $(= 2m)$ agents. Agent i has a quasi-linear expected utility function so that she receives a payoff of

$$u_i^n = x_i v_i^n - p \tag{1}$$

when she obtains an object with probability x_i and pays p in expectation to the seller. The valuation v_i^n takes the following form:

$$v_i^n(s^n) = \alpha_i + \frac{1}{n}\sum_{j=1}^{n} \beta_j, \tag{2}$$

where s^n denotes vector $(s_1, \dots, s_n)$ and each $s_j = (\alpha_j, \beta_j) \in \mathbb{R}^2$ is the signal that agent j observes as her private information. The second component β_i is decomposed as $\beta_i = \theta + \delta_i$, where θ and δ_i are random variables that no agent can directly observe. The random variables α_i, δ_i, and θ are all independently and uniformly distributed on $[0, 1]$.

In this environment, no ex post incentive-compatible mechanism achieves full efficiency. To see this, let α_{-i}^M denote the median of $\alpha_{-i} = (\alpha_j)_{j \neq i}$. (Recall that $n = 2m$ is even, so α_{-i} consists of an odd number of elements and thus some element of α_{-i} coincides with α_{-i}^M.) If $(x_1, \dots, x_n) \in [0,1]^n$ is an efficient allocation, then

$$x_i \begin{cases} = 0 & \text{if } \alpha_i < \alpha_{-i}^M \\ \in [0,1] & \text{if } \alpha_i = \alpha_{-i}^M \\ = 1 & \text{if } \alpha_i > \alpha_{-i}^M. \end{cases} \tag{3}$$

Fix agent i and signals s_{-i}^n such that $\alpha_i^M \in (0,1)$. Let $p_{i,1}$ be agent i's payment when $x_i = 1$, and let $p_{i,0}$ be her payment when $x_i = 0$. (Both $p_{i,1}$ and $p_{i,0}$ may depend on s_{-i}^n but must be independent of s_i^n as a consequence of ex post incentive compatibility.) Let

$$p_i(s_{-i}) = p_{i,1}(s_{-i}^n) - p_{i,0}(s_{-i}^n) \tag{4}$$

denote the difference between the two payments. Consider $\alpha_i^1 = \alpha_{-i}^M + \varepsilon$ and $\alpha_i^2 = \alpha_{-i}^M - \varepsilon$, where ε is a small positive number. Ex post incentive compatibility requires $\alpha_{-i}^1 + \frac{1}{n}\sum_j \beta_j \geq p_i(s_{-i}^n)$ and $\alpha_{-i}^2 + \frac{1}{n}\sum_j \beta_j \leq p_i(s_{-i})$. By taking the limit $\varepsilon \to 0$,

$$p_i(s_{-i}^n) = \alpha_{-i}^M + \frac{1}{n}\sum_{j=1}^{n} \beta_j = \left[\alpha_{-i}^M + \frac{1}{n}\sum_{j \neq i} \beta_j\right] + \frac{\beta_i}{n}. \tag{5}$$

This is a contradiction because $p_i(s_{-i}^n)$ cannot depend on agent i's signal s_i, in particular its second component β_i.

In contrast, full efficiency is achieved with a continuum of agents in *Walrasian equilibria*. In the limit as $m \to \infty$, the valuation function $v_i^n(s^n)$ converges to

$$v^\infty(\alpha_i|\theta) = \alpha_i + \theta + \frac{1}{2}. \tag{6}$$

Because there are twice as many agents as objects and the aggregate distribution of α is uniform on $[0,1]$, the market-clearing price is equal to the median value of v^∞:

$$p^\infty(\theta) = \theta + 1.^{1} \tag{7}$$

In large but finite economies, the Walrasian equilibrium above can be approximated by the following mechanism, which generalizes the *random priority mechanism* (aka *random serial dictatorship*):

[1]Indeed, a game with a continuum of agents with i.i.d. signals is ill-defined because a continuum of i.i.d. random variables easily fails to be measurable (Feldman and Gilles, 1985; Judd, 1985). To avoid this technical problem, without formally modeling such a game, we mechanically define a mechanism as a pair of functions $\varphi^\infty = (x^\infty, t^\infty)$ that map signal s and state θ to the probability $x^\infty(s,\theta)$ of winning and payment $t^\infty(s,\theta)$. This approach suffices for our purpose because we are interested in finite-economy approximation rather than the modeling of infinite economies themselves.

(i) Each agent i reports $\hat{s}_i = (\hat{\alpha}_i, \hat{\beta}_i) \in S \equiv [0,1] \times [0,2]$ simultaneously.

(ii) A priority $\succ$ over the agents is randomly chosen.

(iii) Agent i receives an object and pays

$$p_i^n(\hat{s}_{-i}^n) = \frac{1}{n-1}\sum_{j\neq i}\hat{\beta}_j + \frac{1}{2} \tag{8}$$

whenever

(a) $v_i(\hat{s}^n) > p_i(\hat{s}_{-i}^n)$, and

(b) $m > \max_{s_i'} \sum_{j\succ i} 1\{v_j^n(s_i', \hat{s}_{-i}^n) > p_j^n(s_i', \hat{s}_{-\{i,j\}}^n)\}$.

(Otherwise, agent i does not receive an object and pays nothing.)

Let us explain the idea behind step (iii) in more detail. The price $p_i(\hat{s}_{-i})$ is an estimate of the full-information market clearing price that does not involve agent i's signal to preserve incentive compatibility.[2] Condition (a) simply ensures that it is individually rational for agent i to buy an object at the price $p_i^n(\hat{s}_{-i}^n)$. Finally, condition (b) ensures feasibility, while at the same time making sure that ex post incentive compatibility is preserved. A seemingly more straightforward feasibility condition would say that there are enough objects remaining to get one to agent i:

$$m > \sum_{j\succ i} 1\{v_j^n(\hat{s}^n) > p_j^n(\hat{s}_{-j}^n)\}. \tag{9}$$

This condition, however, may allow agents to profitably deviate from truth-telling: Agents may be able to prevent other agents from getting an object by manipulating their reports. To prevent such a manipulation, condition (b) must be independent of agent i's own message $\hat{s}_i$.

The following simple two-agent situation exemplifies that the alternative condition (9) induces a profitable deviation. Suppose $s_1 = s_2 = (0.6, 0.4)$ so that $v_1(s_1, s_2) = v_2(s_1, s_2) = 1$ and $p_1(s_2) = p_2(s_1) = 0.9$. With the truthful reports, the object is sold to the agent with the higher priority for price 0.9. Consider the deviation that agent 1 submits a false report $\hat{s}_1 = (0, 2)$. This report prevents agent 2 from winning because $v_2(\hat{s}_1, s_2) = 1.3 < 2 = p_2(\hat{s}_1)$. In contrast, it does not change agent 1's estimated valuation, $v_1(\hat{s}_1, s_2) = 1$, or personalized price for the object, $p_1(s_2) = 0.9$. The outcome with this report is such that agent 1 can buy the object for price 0.9 for sure and the agent prefers this outcome to the truth-telling one.

It is intuitive that the generalized random priority mechanism is asymptotically efficient because all the prices and valuations appearing in step (iii) uniformly converge to their

[2] Note that $\sum_{j\neq i}\beta_j$ converges to $\theta + 1/2$ and hence $p_i^n(s_{-i}^n)$ converges to $p^\infty(\theta) = \theta + 1$.

infinite-economy counterparts. To see this, let $\tilde{s}_i$ and $\tilde{\theta}$ denote the random variables that represent the signal of agent i and the state. Consider $p_i^n(\tilde{s}_{-i}^n)$ and $p_j^n(s_i', \tilde{s}_{-\{i,j\}}^n)$, the prices appearing in conditions (a) and (b) with the reports $\hat{s}^n$ replaced by the true signals $\tilde{s}^n$. We focus on the latter because it can represent the former as a special case by interchanging i and j. All of these prices are uniformly close enough to the best unbiased estimator of the true state $\tilde{\theta}$, namely $p^n(\tilde{s}^n) = \frac{1}{n}\sum \hat{\beta}_j + \frac{1}{2}$, which consistently estimates $p^\infty(\tilde{\theta})$, that is,

$$
\begin{aligned}
\max_{i,j,s_i'} \left| p_j^n(s_i', \tilde{s}_{-\{i,j\}}^n) - p^\infty(\tilde{\theta}) \right| &\leq \max_{i,j,s_i'} \left| p_j^n(s_i', \tilde{s}_{-\{i,j\}}^n) - p^n(\tilde{s}^n) \right| + \left| p^n(\tilde{s}^n) - p^\infty(\tilde{\theta}) \right| && (10)\\
&\leq \max_{i,j,s_i'} \left[\frac{1}{n} \cdot \tilde{\beta}_j + \frac{1}{n(n-1)} \sum_{k \neq j} \tilde{\beta}_k \right] + \left| p^n(\tilde{s}^n) - p^\infty(\tilde{\theta}) \right| && (11)\\
&\to_p 0, && (12)
\end{aligned}
$$

where the arrow $\to_p$ signifies the convergence in probability. Similarly, the valuation $v_j(s_i', \tilde{s}_{-i})$ also uniformly converges to $v^\infty(\tilde{\alpha}_j|\tilde{\theta})$ in probability. These two facts suggest that condition (a) also "uniformly converges" to the infinite-economy assignment rule:

$$
v^\infty(\tilde{\alpha}_i|\theta) > p^\infty(\theta). \tag{13}
$$

They also imply that the rationing rule, condition (b), virtually disappears as m goes to ∞, because its right-hand side is asymptotically bounded by m:

$$
\begin{aligned}
&\frac{1}{m} \cdot \max_{s_i' \in S} \sum_{j \succ i} 1\{v_j^n(s_i', \hat{s}_{-i}^n) > p_j^n(s_i', \hat{s}_{-\{i,j\}}^n)\} \\
&\leq \frac{2}{n} \cdot \max_{s_i' \in S} \sum_{j=1}^{n} 1\{v_j^n(s_i', \hat{s}_{-i}^n) > p_j^n(s_i', \hat{s}_{-\{i,j\}}^n)\} && (14)\\
&\to_p 2 \cdot \int 1\{v^\infty(\alpha|\tilde{\theta}) > p^\infty(\tilde{\theta})\} \mathrm{d}\alpha = \frac{1}{2} \cdot 2 = 1. && (15)
\end{aligned}
$$

More formal arguments are presented in subsequent sections with the general model presented in the next section.

3 The Model

3.1 Environment

We consider a sequence of finite economies indexed by the number of agents, $n \in \mathbb{N} \equiv \{1, 2, 3, \ldots\}$. In finite economies, each agent $i \in N_n = \{1, \ldots, n\}$ receives a *signal* $s_i \in S \equiv [0,1]^{d_S}$, independently and identically distributed with density $f(s_i|\theta)$ conditional on *state* $\theta \in \Theta \equiv [0,1]^{d_\Theta}$.[3] The density $f(s|\theta)$ is positive and continuous on $S \times \Theta$, and any two different states $\theta, \theta' \in \Theta$ are statistically distinguishable, i.e., $f(s|\theta) \neq f(s|\theta')$ for some $s \in S$. The prior on θ is given by a positive continuous density $f(\theta)$. The conditional density function of θ given $s^n = (s_1, \ldots, s_n)$ is denoted by $f(\theta|s^n)$. Explicitly,

$$f(\theta|s^n) = \frac{\prod_{i=1}^n f(s_i|\theta) f(\theta)}{\int_\Theta \prod_{i=1}^n f(s_i|\theta') f(\theta') \mathrm{d}\theta}. \tag{16}$$

Throughout the paper, we use symbols s_i and θ to denote the (deterministic) elements of S and Θ, respectively, whereas the corresponding random variables are denoted by $\tilde{s}_i$ and $\tilde{\theta}$.

There are L goods. The per-capita supply of each good ℓ is $q_\ell \in (0, \infty]$, meaning that the total supply of commodity ℓ is $\lfloor n \cdot q_\ell \rfloor$. The set of all possible *consumption bundles* is $X \subseteq \{0, 1, 2, \ldots, \bar{x}\}^L$, where a positive integer $\bar{x}$ signifies the maximum consumption level. The set X contains the zero vector, 0.

Each agent i has a common quasi-linear expected utility function

$$u^\infty(x_i, t_i|s_i, \theta) = v^\infty(x_i|s_i, \theta) - t_i. \tag{17}$$

Here, $x_i \in X$ is a consumption bundle, $t_i \in \mathbb{R}$ is the amount of payment, $s_i \in S$ is the signal observed by agent i, and $\theta \in \Theta$ is a state, which serves as a common-value component.[4] The function $v^\infty(x|s, \theta)$ is continuous with respect to s and θ. Without loss of generality, we normalize v^∞ so that $v^\infty(0|s, \theta) = 0$ and $v^\infty(x|s, \theta) \leq 1$ for all (x, s, θ).[5]

We impose the following assumption on v^∞.

Assumption 1. *For each $x_i, x_i' \in X$ and $\theta \in \Theta$, mapping*

$$w_{x_i, x_i', \theta}(s_i) = v(x_i|s_i, \theta) - v(x_i'|s_i, \theta) \tag{18}$$

is continuously differentiable, and its first-order derivative is nonzero for all $s_i \in S$.

We evaluate the welfare of agents conditional on signals s^n rather than on state θ because the vector s^n contains all the available information in the economy although θ is the

[3] More generally, S and Θ can be compact subsets of Euclidean spaces such that (i) S has a nonempty interior whose closure coincides with S and (ii) either Θ has a positive Lebesgue measure or Θ is a countable set. The function $f(\theta)$ is a continuous density when Θ has a positive measure. When Θ is countable, $f(\theta)$ is the probability of θ. In either case, $f(\theta) > 0$ for all $\theta \in \Theta$. The results and proofs extend to the general case without modification.

[4] The signal s_i may contain the characteristics of the agent so that agent i's utility function $u^\infty(x_i, t_i|s_i, \theta)$ may depend on s_i.

[5] Such v^∞ can be constructed from the original function v_o^∞ by

$$v^\infty(x_i|s_i, \theta) = \frac{v_o^\infty(x_i|s_i, \theta) - v_o^\infty(0|s_i, \theta)}{\max\{1, \max_{x_i, s_i, \theta}[v_o^\infty(x_i|s_i, \theta) - v_o^\infty(0|s_i, \theta)]\}}.$$

fundamental parameter that nobody ever observes. Define $u_i^n(x_i, t_i|s^n)$ and $v_i^n(x_i|s^n)$ as the expected values of $u^\infty(x_i, t_i|s_i)$ and $v^\infty(x_i|s_i)$, respectively, conditional on s^n. That is,

$$v_i^n(x_i|s^n) = \int_\Theta v^\infty(x_i|s_i, \theta) f(\theta|s^n) \mathrm{d}\theta \tag{19}$$

and $u_i^n(x, t|s^n) = v_i^n(x|s^n) - t$.

We can naturally extend the domains of $u^\infty(\cdot|s, \theta)$, $v^\infty(\cdot|s, \theta)$, $u_i^n(\cdot|s^n)$, and $v_i^n(\cdot|s^n)$ to lottery spaces. For example, $u_i^n(\lambda|s^n)$ is defined by

$$u_i^n(\lambda|s^n) = \int_{(X\times\mathbb{R})^n} u_i^n(x_i, t_i|s^n) \mathrm{d}\lambda(x_1, t_1, \dots, x_n, t_n) \tag{20}$$

for $\lambda \in \Delta((X \times \mathbb{R})^n)$, where $\Delta(A)$ denotes the set of lotteries on A.

3.2 Mechanisms

3.2.1 Mechanisms for Finite Economies

We first define mechanisms for finite economies and their properties. A *mechanism* is a sequence $\varphi = \{\varphi^n\}$ of mappings $\varphi^n : S^n \to \Delta((X \times \mathbb{R})^n)$. We say φ is *money-free* if

$$\varphi^n(s^n)\big\{(x_1, 0; \dots; x_n, 0) : x_1, \dots, x_n \in X\big\} = 1 \tag{21}$$

for all n and $s^n \in S^n$; i.e., the outcomes of the mechanism never involve monetary transfers.

Mechanism $\{\varphi^n\}$ is *ex post incentive compatible* if

$$u_i^n\big(\varphi^n(s^n); s^n\big) \geq u_i^n\big(\varphi^n(\hat{s}_i, s_{-i}^n); s^n\big) \tag{22}$$

for all n, $i \in N_n$, $s^n \in S^n$, and $\hat{s}_i \in S$.

We define the (approximate) feasibility of allocations and then mechanisms. Allocation $x^n \in X^n$ is *ε-feasible* if $x_1 + \cdots + x_n \leq (1+\varepsilon)nq$. Mechanism $\{\varphi^n\}$ is *ε-feasible* if $\varphi^n(s^n)$ assigns probability 1 on ε-feasible $x^n \in X^n$ for all $n \in \mathbb{N}$ and $s^n \in S^n$. When an allocation or a mechanism is 0-feasible, we simply say that it is *feasible*.

3.2.2 Mechanisms for the Infinite Economy

We now define mechanisms for the infinite economy, which can be viewed as the limit of finite economies. The infinite economy has a continuum of agents whose signals are distributed according to the density $f(s|\theta)$. Conditional on state θ, the economy has no aggregate uncertainty. This naturally motivates us to focus on the class of mechanisms that use only state θ and agent i's report s_i to determine agent i's assignment.

Formally, an *∞-mechanism* is a mapping $\varphi^\infty = (z^\infty, t^\infty) : S \times \Theta \to \Delta(X) \times \mathbb{R}$.[6] An ∞-mechanism $\varphi^\infty = (z^\infty, t^\infty)$ is *money-free* if $t^\infty(s|\theta) = 0$ for all $s \in S$ and $\theta \in \Theta$.

[6]An alternative definition is $\varphi^\infty : S \times \Theta \to \Delta(X \times R)$, but this definition is functionally equivalent to our first definition by taking the expectation of the monetary part.

An ∞-mechanism φ^∞ is *incentive compatible* if

$$u^\infty\big(\varphi^\infty(s|\theta)\big|s,\theta\big) \geq u^\infty\big(\varphi^\infty(\hat{s}|\theta)\big|s,\theta\big) \tag{23}$$

for all $s, \hat{s} \in S$, and $\theta \in \Theta$.

Allocation (measurable mapping) $z : S \to \Delta(X)$ is *ε-feasible* at $\theta \in \Theta$ if

$$\int \mathbb{E}_X[z(s)] f(s|\theta)\mathrm{d}s \leq (1+\varepsilon)q, \tag{24}$$

where $\mathbb{E}_X[z'] = \sum_{x\in X} x \cdot z'(x)$ for each $z' \in \Delta(X)$. An ∞-mechanism $\varphi^\infty = (z^\infty, t^\infty)$ is *ε-feasible* if $z^\infty(\cdot|\theta)$ is ε-feasible at θ for all $\theta \in \Theta$. Again, we define a *feasible* ∞-mechanism as a 0-feasible one.

Next, we introduce the notion of a budget set, a generalization of Walrasian budget sets, together with related concepts. Budget sets are closely related to incentive compatibility, as we will see in the next paragraph. A *budget set* B is a set of pairs (z,t) of a lottery $z \in \Delta(X)$ on X and payment $t \in \mathbb{R}$ such that $0 \in B$. Budget set B is *money-free* if $t = 0$ for all $(z,t) \in B$; i.e., no option in B involves a monetary transfer. Let $\mathcal{B}$ be the space of budget sets and $\mathcal{B}_f$ the set of finite budget sets. A *budget rule* is a mapping $B^\infty : \Theta \to \mathcal{B}$, and it is *finite* if $B^\infty(\theta) \in \mathcal{B}_f$ for all $\theta \in \Theta$. A budget rule B^∞ is *money-free* if $B^\infty(\theta)$ is money-free for all $\theta \in \Theta$.

Budget sets naturally represent incentive-compatible ∞-mechanisms. A budget rule B^∞ *implements* an ∞-mechanism φ^∞ if

$$\varphi^\infty(s|\theta) \in \underset{z\in B^\infty(\theta)}{\arg\max}\, u^\infty(z|s,\theta). \tag{25}$$

We simply say that $(\varphi^\infty, B^\infty)$ is an ∞-mechanism when B^∞ implements φ^∞, slightly abusing notations. Note that a mechanism φ^∞ can be implemented only by some B^∞ when φ^∞ is incentive compatible. Conversely, an incentive-compatible ∞-mechanism φ^∞ always has a budget set rule $B^\infty(\theta) = \{\varphi^\infty(s|\theta) : s \in S\}$ that implements φ^∞. (However, the induced budget set as above may not satisfy some desirable properties. We thus also consider other implementations of incentive compatible mechanisms later in this paper.)

Example 1. Consider a limit economy with a single good and unit demands so that $X = \{0,1\}$. When $q = 1$, selling the good at a fixed price p is a feasible, incentive-compatible ∞-mechanism with a finite budget set rule. That is, a finite budget set rule $B^\infty = \{(0,0),(1,p)\}$ implements a feasible ∞-mechanism

$$\varphi^\infty(s|\theta) = \begin{cases} (1,p) & \text{if } v^\infty(1|s,\theta) \geq p \\ (0,0) & \text{otherwise.} \end{cases} \tag{26}$$

Example 2. Consider a limit economy with two goods and unit demands so that $X = \{(0,0),(1,0),(0,1)\}$. Let (z_1,z_2) denote the lottery in which an agent obtains good ℓ with probability z_ℓ. Consider a money-free budget set $B^\infty = \{(0,0),(1/2,0),(0,1/2)\}$.[7] That is,

[7]More precisely, this budget set should be written as $\{((0,0),0),((1/2,0),0),((0,1/2),0)\}$.

agents may choose a 50% chance of winning good 1, a 50% chance of winning good 2, or opting out. This budget set (rule) B^∞ implements a nonmonetary ∞-mechanism

$$\varphi^\infty(s|\theta) = \begin{cases} (1/2, 0) & \text{if } v^\infty((1/2,0)|s,\theta) > \max\{0, v^\infty((0,1/2)|s,\theta)\} \\ (0, 1/2) & \text{if } v^\infty((0,1/2)|s,\theta) > \max\{0, v^\infty((1/2,0)|s,\theta)\} \\ (0,0) & \text{otherwise,} \end{cases} \tag{27}$$

which is feasible when $q_1, q_2 \geq 1/2$.

4 Generalized Random Priority Mechanism with Budgets

We introduce a new class of mechanisms, the *generalized random priority mechanism with budgets* (GRP), which generalizes the random priority mechanism (aka random serial dictatorship). Any mechanism in this class is feasible and ex post incentive compatible. GRP may have other desirable properties, such as asymptotic efficiency, depending on the specification of the rule.

GRP is based on budget sets; the rule of GRP depends on a sequence $B^n = \{B_i^n\}_{i=1}^n$ of mappings that determines assignment of budget sets. Each B_i^n is a mapping from a vector of signals $s_{-i}^n \in S^{n-1}$ to a finite budget set $B_i^n(s_{-i}^n) \in \mathcal{B}_f$. Given such B^n, GRP, denoted by GRP$[B^n]$, runs as follows:

(i) The mechanism randomly chooses a *priority* $\succ$ over agents with an equal probability. Independently of $\succ$, the mechanism assigns a number r_i to each agent $i \in N_n$ so that r_i is independently and uniformly distributed on $[0,1]$. This r_i is a randomization device used to determine the outcome of a lottery.

(ii) Each agent $i \in N_n$ simultaneously submits a *report* $\hat{s}_i \in S$. From $\hat{s}^n$, the mechanism assigns agent i a finite budget set $B_i^n(\hat{s}_{-i}^n)$, independent of i's own message, $\hat{s}_i$.

(iii) The mechanism sequentially constructs allocations and transfers, $(x_i(\succ; r_{\succeq i}; \hat{s}^n), t_i(\succ; r_{\succeq i}; \hat{s}^n))$, from the top of the priority ordering $\succ$ as follows:[8]

(a) As a proxy of agent i, the mechanism chooses $(z_i(\succ; r_{\succ i}; \hat{s}^n), t_i(\succ; r_{\succ i}; \hat{s}^n))$ from

[8]Here, $r_{\succeq i} = (r_j)_{j \succeq i}$ and $r_{\succ i} = (r_j)_{j \succ i}$.

the solutions of the following problem:

$$\begin{aligned}
&\max_{z,t} \; u_i^n(z,t|\hat{s}^n) \\
&\text{subject to (i)} \quad (z,t) \in B_i^n(\hat{s}_{-i}^n) \quad \text{and} && (28) \\
&\text{(ii)} \;\; x \le \lfloor nq \rfloor - \max_{s_i' \in S} \sum_{j \succ i} x_j(\succ, r_{\succeq j}; s_i', \hat{s}_{-i}^n) && (29) \\
&\text{for all } x \in X \text{ s.t. } z(x) > 0.
\end{aligned}$$

That is, condition (ii) ensures the feasibility of lottery z, no matter which signal agent i reports and which outcome x is realized from the lottery z. If this problem has multiple solutions, the mechanism uses a deterministic predetermined rule to select (z_i, t_i) from them.[9]

(b) An outcome $x_i(\succ, r_{\succeq i}; \hat{s}^n)$ is realized from the lottery $z_i(\succ, r_{\succ i}; \hat{s}^n)$ by using i's randomization device r_i in the following manner:

$$x_i(\succ, r_{\succeq i}; \hat{s}^n) = x^{(k)} \quad \text{if } r \in (R_{k-1}, R_k], \tag{30}$$

where $R_k = \sum_{k' \le k} z_i(x^{(k')}|\succ, r_{\succ i}; \hat{s}^n)$. Here, $(x^{(1)}, \dots, x^{(|X|)})$ is the lexicograhical order of X.[10]

Given the name of this mechanism, we wish to make sure that GRP does in fact generalize the random priority mechanism. The example below shows that it is indeed the case.

Example 3 (Random Priority)**.** Suppose that values are private: $v(x|s_i, \theta) = v(x|s_i)$. Let the budget set be constant: $B_i^n = X \times \{0\}$. Then, the GRP mechanism works as follows:

(i) A priority $\succ$ is randomly chosen.

[9] Although such a rule is arbitrary, for concreteness, we specify the rule we use. The mechanism selects the smallest element as (z_i, t_i) from the solution set with respect to a linear order $\gtrdot$ on $\Delta(X) \times \mathbb{R}$. Here, $\gtrdot$ is the lexicographical order on $\Delta(A) \times \mathbb{R}$ by identifying $(z,t) \in \Delta(A) \times \mathbb{R}$ with a vector $(z(x^{(1)}), \dots, z(x^{(|X|)}), t)$.

[10] For example, suppose $z_i(\succ, r_{\succ i}; \hat{s}^n)$ is the lottery such that

$$(z(0,0), z(0,1), z(1,0), z(1,1)) = (0.1, 0.4, 0.3, 0.2).$$

Then, as a function of r_i, the outcome $x_i(\succ, r_{\succeq i}; \hat{s}^n)$ is given by

$$x_i(\succ, r_{\succeq i}; \hat{s}^n) = \begin{cases} (0,0) & \text{if } r_i \in (0, 0.1] \\ (0,1) & \text{if } r_i \in (0.1, 0.5] \\ (1,0) & \text{if } r_i \in (0.5, 0.8] \\ (1,1) & \text{if } r_i \in (0.8, 1] \end{cases}$$

(ii) Agents simultaneously report their signals s_i (or equivalently, their utility functions $v_i(x) = v(x|s_i)$).

(iii) Agent i obtains $x_i(\succ; v_j)$, which is a solution of the problem: $\max v_i(x)$ subject to the feasibility constraint $x \leq \lfloor nq \rfloor - \sum_{j \succ i} x_j(\succ; v_j)$.

The GRP mechanism is feasible by construction. Thanks to the feasibility constraint (30), the lottery $z_i(\succ, r_{\succ i}; \hat{s}^n)$ never assigns a positive probability to an assignemt x that may violate the inequality $x + \sum_{j \succ i} x_j(\succ, r_{\succeq j}; \hat{s}^n) \leq \lfloor nq \rfloor$, and hence $\sum_{j \succeq i} x_j(\succ, r_{\succeq j}; \hat{s}^n) \leq \lfloor nq \rfloor$. In particular, i can be chosen as the last agent. Therefore, the feasibility inequality

$$\sum_{j \in N_n} x_j(\succ, r_{\succeq j}; \hat{s}^n) \leq \lfloor nq \rfloor \tag{31}$$

is satisfied for all $\succ$ and r^n.

It is also straightforward that this mechanism is ex post incentive compatible. The key idea is, just like in the example in Section 2, that both constraint (30) and the budget set $B_i^n(\hat{s}_{-i}^n)$ are independent of $\hat{s}_i$. Therefore, agent i cannot manipulate the set of choices. For any realization of $(\succ, r_{\succ i})$, agent i who misreported would still have the same set of choices, but the mechanism might then choose a suboptimal choice from this set. By integrating out $(\succ, r_{\succ i})$,

$$u_i^n\big(\mathrm{GRP}[B^n](s^n)\big|s^n\big) = \frac{1}{n!}\sum_{\succ}\int u_i^n\big(z_i(\succ, r_{\succ i}; s^n)\big|s^n\big)\mathrm{d}r_{\succ i} \tag{32}$$

$$\geq \frac{1}{n!}\sum_{\succ}\int u_i^n\big(z_i(\succ, r_{\succ i}; s_i', s_{-i}^n)\big|s^n\big)\mathrm{d}r_{\succ i} \tag{33}$$

$$= u_i^n\big(\mathrm{GRP}[B^n](s_i', s_{-i}^n)\big|s^n\big). \tag{34}$$

This is the definition of ex post incentive compatibility.

Theorem 1 summarizes the above arguments.

Theorem 1. *For any choice of $B = \{B_i^n\}$, the generalized random priority mechanism* $\mathrm{GRP}[B]$ *is feasible and ex post incentive compatible.*

5 Main Results: Approximation Theorems

This section presents the main results of the paper. These results show how any ∞-mechanism $(\varphi^\infty, B^\infty)$ for the infinite market can be approximated by an appropriately constructed GRP mechanism for large finite markets. The first result, Theorem 2, follows a "point-wise" approach in the sense that the constructed mechanism approximates φ^∞ in each θ. This theorem has a limitation that the approximation is guaranteed only in states

θ at which B^∞ is continuous.[11] The second result, Theorem 3, overcomes this limitation by using an additional step (discretization of the state space Θ) in the construction of the approximating mechanism.

Theorem 2. *Let $(\varphi^\infty, B^\infty)$ be an α-feasible incentive-compatible ∞-mechanism such that the budget rule B^∞ is finite. Suppose for a subset $\Theta_{\mathrm{C}} \subseteq \Theta$ with a positive Lebesgue measure that for all states $\theta \in \Theta_{\mathrm{C}}$, $B^\infty(\cdot)$ is continuous at θ. Then there exists budget rule $\{B^n\}_{n=2}^\infty$ for finite economies such that for all $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$, the following hold with a probability more that $1 - \varepsilon$ conditional on $\tilde{\theta} \in \Theta_{\mathrm{C}}$:*

$$\max_{i \in N_n} \left| u_i^n(\mathrm{GRP}[B^n](\tilde{s}^n)|\tilde{s}^n) - u^\infty(\varphi^\infty(\tilde{s}_i|\tilde{\theta})|\tilde{s}_i, \tilde{\theta}) \right| < \alpha + \varepsilon, \tag{35}$$

$$\left| \mathrm{Rev}^{\mathrm{GRP}}(\tilde{s}^n) - \mathrm{Rev}^\infty(\tilde{\theta}) \right| < 2t_*(\alpha + \varepsilon) + \varepsilon. \tag{36}$$

Here,

$$\mathrm{Rev}^{\mathrm{GRP}}(s^n) = \frac{1}{n} \int (t_1 + \cdots + t_n)\, \mathrm{dGRP}[B^n](s^n)(x_1, t_1, \ldots, x_n, t_n), \tag{37}$$

$$\mathrm{Rev}^\infty(\theta) = \int t^\infty(s|\theta) f(s|\theta) \mathrm{d}s, \tag{38}$$

and $t_* = \sup(\{-t : (\lambda, t) \in B^\infty(\Theta)\} \cup \{1\})$ is an upperbound on the payments any agent can receive from φ^∞. If φ^∞ is money-free, $\{\mathrm{GRP}[B^n]\}_{n=2}^\infty$ is also money-free.

Proof. See Appendix A.3. □

Conditioning on $\tilde{\theta} \in \Theta_{\mathrm{C}}$, Theorem 2 asserts that two approximation conditions, (35) and (36), hold with a probability almost equal to 1.[12] The first condition, (35), gives approximation of payoffs: The payoff that GRP yields to agent i is close to the payoff that φ^∞ would yield to her if she were a part of the infinite economy with the signal $\tilde{s}_i$. The second condition, (36), states that the revenue from GRP is almost equal to the revenue from φ^∞, provided t_*, the largest payment that an agent can receive, is finite. This requirement on t_* is innocuous in applications in which the mechanism never pays money to agents (and thus $t_* = 1$).

Theorem 2 allows α-feasibility so that the theorem can be used in the proof of the second result, Theorem 3, where slightly infeasible ∞-mechanisms are potentially considered. The

[11] A correspondence is continuous at a point if it is both upper and lower hemi-continuous. This notion of continuity is metricized by the Hausdorff metric, which is indeed used in our proofs.

[12] Formally, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the random variables $\tilde{s}_1$, $\tilde{s}_2$, …, and $\tilde{\theta}$ are defined. These random variables are infinitely many, but in an n-agent economy only $\tilde{s}_1$, …, $\tilde{s}_n$, and $\tilde{\theta}$ are considered. When we claim that *a statement Y occurs with probability more than p*, it means that there exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) > p$ and Y holds whenever $\omega \in A$. By doing so, we avoid the measure-theoretical cumbrance that the set $\{\omega \in \Omega : Y \text{ holds}\}$ may not be measurable.

α-feasibility prevents the approximation errors, $\alpha + \varepsilon$ and $2t_*(\alpha + \varepsilon) + \varepsilon$, from converging to 0, but these errors disappear as α decreases to 0 (as seen in Theorem 3).

The proof of Theorem 2 is constructive. First, from reports $\hat{s}^n$, the GRP mechanism constructs a maximum likelihood estimator of the true state $\tilde{\theta}$ for each agent i, based only on the reports of the other agents:

$$\theta_i^{\mathrm{MLE}}(\hat{s}_{-i}^n) \in \arg\max_{\theta \in \Theta} \Big(\prod\nolimits_{j \in N_n \setminus \{i\}} f(s_j|\theta) \Big). \tag{39}$$

Next, the mechanism assigns to agent i a budget set

$$B_i^n(\hat{s}_{-i}^n) = B^\infty(\theta_i^{\mathrm{MLE}}(\hat{s}_{-i}^n)), \tag{40}$$

i.e., the budget set that agent i would have if she were in the infinite economy in state θ_i^{MLE}. In the rest of the proof, we show that this GRP mechanism has the properties stated in the theorem.

The proof would be simple if we could ignore the feasibility constraint (30) appearing in the definition of GRP. A rough sketch of the proof is as follows. By the consistency of maximum likelihood estimators, $\{\theta_i^{\mathrm{MLE}}\}_{i=1}^n$ uniformly converge to the true state $\tilde{\theta}$ as $n \to \infty$. The budget sets $\{B_i^n(\tilde{s}_{-i}^n)\}_{i=1}^n$ also uniformly converge to the limit budget $B^\infty(\tilde{\theta})$ because B^∞ is continuous at $\tilde{\theta}$. Since B_i^n is close to $B^\infty(\tilde{\theta})$, agent i may choose $(z_i^n, t_i^n) \in B_i^n$ that is almost the same as $\varphi^\infty(\tilde{s}_i|\tilde{\theta})$; the latter is what agent i would choose if she were present in the limit economy with budget set $B^\infty(\tilde{\theta})$. Some agents may find a better choice than (z_i^n, t_i^n). However, their fraction must be small, and they cannot significantly increase their utility levels because all the agents have almost the same preferences as the limit ones, $u^\infty(\cdot|s, \theta)$.

The feasibility constraint (30) adds three difficulties. The first is the rationing effect: It is unclear how many agents can choose $(z_i^n, t_i^n) \in B_i^n$, an option very close to the limit choice, $\varphi^\infty(\tilde{s}_i|\tilde{\theta})$, even when B^∞ has no lotteries. The second is due to the supremum: x_j used in the constraint may significantly differ from the actual assignment. The third is the independent realization of lotteries. For instance, suppose that each of $2m$ agents has a 50% chance of winning one unit of a good whose supply is m. If all the lotteries are independent, infeasibility is unavoidable; e.g., when $m = 1$, there is a 25% chance that both agents will win. However, this infeasibility turns out not to play an important role when m is large: The probability that the number of winners is less than $(1 + \varepsilon)m$ converges to 1 as $m \to \infty$.

Flexible Approximation Theorem: Proxy Approach

A limitation of Theorem 2 is that the approximation is guaranteed only in states θ at which B^∞ is continuous. Thus, to be sure that the approximation holds without conditioning, it is in principle sufficient to show that the budget rule B^∞ is continuous almost everywhere. Proving the latter, however, would require development of a theory of "regular economies" for φ^∞, which might enable us to find a "price path" B^∞ that is locally continuous except for measure-0 "irregular" states θ, similar to the corresponding construction in the Walrasian equilibrium theory. Moreover, even in the context of Walrasian equilibria, the theory of regular economies needs certain regularity on the mapping from the parameters to excess demand functions (Mas-Colell, 1990, 5.8.12 and 5.8.18). It is unclear that such a regularity condition would always hold for the class of models considered in the current paper.

We take an alternative approach. The second approximation result, Theorem 3, only needs finiteness of the budget rule B^∞. The key idea is discretization: The state space Θ is partitioned into finite sets $\{\Theta_k\}$, each of which has a representative state θ_k^P that serves as a *proxy* of all the states $\theta_k \in \Theta_k$. We use $B^\infty(\theta_k^P)$ instead of $B^\infty(\theta_k)$, ensuring that the budget set rule $\theta_k \mapsto B^\infty(\theta_k^P)$ is continuous almost everywhere. As we explain below, the finer the discretization, the smaller are the approximation errors arising from it. Thus, in the proof, we shrink the diameter of each Θ_k shrinks to 0 as $n \to \infty$. Formally, we define a *proxy rule* as a sequence $\{P^n\}_{n=1}^\infty$ of measurable mappings $P^n : \Theta \to \Theta$ such that

$$\mathfrak{m}\{\theta \in \Theta : P^n \text{ is not continuous at } \theta\} = 0 \tag{41}$$

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} \|P^n(\theta) - \theta\| = 0, \tag{42}$$

where $\mathfrak{m}$ is the Lebesgue measure.

Theorem 3. *Let* $(\varphi^\infty, B^\infty)$ *be a feasible incentive-compatible* ∞*-mechanism such that* B^∞ *is finite. There exist* $\{B^n\}_{n=1}^\infty$ *and a proxy rule* $\{P^n\}_{n=1}^\infty$ *such that for all* $\varepsilon > 0$*, there exists* $N > 0$ *such that for all integers* $n \geq N$*, the inner probability of the following is more than* $1 - \varepsilon$*:*

$$\max_{i\in N_n} \left| u_i^n(\mathrm{GRP}[B^n](\tilde{s}^n)|\tilde{s}^n) - u^\infty(\varphi^\infty(\tilde{s}_i|\tilde{\theta})|\tilde{s}_i, \theta^P)\right| < \varepsilon, \tag{43}$$

$$\left|\mathrm{Rev}^{\mathrm{GRP}}(\tilde{s}^n) - \mathrm{Rev}^\infty(\theta^P)\right| < t_*\varepsilon, \tag{44}$$

where $\theta^P = P^n(\tilde{\theta})$ *and* $\mathrm{Rev}^{\mathrm{GRP}}$, Rev^∞*, and* t_* *are as defined in Theorem 2. If* φ^∞ *is money-free,* $\{\mathrm{GRP}[B^n]\}$ *is also money-free.*

Unlike Theorem 2, Theorem 3 uses the proxy $\theta^P = P^n(\tilde{\theta})$ instead of the true state $\tilde{\theta}$. Although the proxy utility function $u^\infty(x|\tilde{s}_i, \theta^P)$ can differ from the true one $u^\infty(x|\tilde{s}_i, \tilde{\theta})$, the difference ultimately disappears as n goes to ∞.

In the proof, we apply Theorem 2 to descretized versions of the ∞-mechanism φ^∞ instead of attempting to directly approximate φ^∞ itself. As above, let $\mathcal{P} = \{\Theta_k\}$ be a partition of the state space Θ accompanied with representative elements $\theta_k^P \in \Theta_k$. From this partition, we construct a budget rule $B_\mathcal{P}^\infty$ that is constant on each Θ_k:

$$B_\mathcal{P}^\infty(\theta) = B^\infty(\theta_k^P) \quad \text{if } \theta \in \Theta_k. \tag{45}$$

We consider an ∞-mechanism $\varphi_\mathcal{P}^\infty$ that $B_\mathcal{P}^\infty$ implements. Indeed, such $\varphi_\mathcal{P}^\infty$ is unique up to measure-0 set signals; Lemma 1 below shows that indifference may occur only in a measure-0 set. The mechanism $\varphi^\infty_{\mathcal{P}}$ may not be feasible; its feasibility is guaranteed only at $\theta^P_1, \dots, \theta^P_K$. However, by appropriately specifying the partition $\{\Theta_k\}$, the size of its infeasibility can be arbitrarily small. Recall that Theorem 2 translated the size α of infeasibility into proportional approximation errors for large n. Therefore, by carefully designing the partition $\{\Theta_k\}$, we can also make the errors in Theorem 2 arbitrarily small.

The above argument is formalized in the proof as follows. We construct a sequence of partitions $\mathcal{P}_1, \mathcal{P}_2, \dots$ so that the size of infeasibility in $\varphi^\infty_{\mathcal{P}_k}$ disappears as $k \to \infty$. We apply Theorem 2 to each $(\varphi^\infty_{\mathcal{P}_k}, B^\infty_{\mathcal{P}_k})$ and carefully let k depend on n to ensure that approximation errors in finite economies appropriately disappear in the limit $n \to \infty$.

6 Applications

This section presents two applications of Theorem 3: combinatorial auctions (Section 6.2) and (potentially) multi-unit assignment without monetary transfers (Section 6.3). To evaluate mechanisms presented in these sections, we first introduce concepts regarding fairness and efficiency.

6.1 Fairness and Efficiency

In both applications, we consider the concept of envy-freeness. In finite markets, a feasible random assignment $\lambda \in \Delta((X \times \mathbb{R})^n)$ is *ε-envy-free* at $s^n \in S^n$ if

$$u^n_i(z_i, \tau_i | s^n) \geq u^n_i(z_j, \tau_j | s^n) - \varepsilon \tag{46}$$

for all $i, j \in N_n$, where $(z_1, \tau_1, \dots, z_n, \tau_n)$ are the marginal distribution of λ; i.e., z_i and τ_i are the distributions of agent i's assignment x_i and payment t_i, respectively, in the random assignment λ. A mechanism $\{\varphi^n\}$ is *asymptotically envy-free* if for all $\varepsilon > 0$, there exists N such that $n \geq N$ implies that $\varphi^n(\tilde{s}^n)$ is ε-envy-free with a probability more than $1 - \varepsilon$.

In Section 6.2, we consider an environment with monetary transfers, and thus Pareto efficiency coincides with utilitarian surplus maximization. In finite markets, a feasible random assignment $\lambda \in \Delta((X \times \mathbb{R})^n)$ is *ε-surplus maximizing* at $s^n \in S^n$ if for all feasible random assignments $\hat{\lambda} \in \Delta((X \times \mathbb{R})^n)$,

$$\frac{1}{n} \sum_{i=1}^n v^n_i(z_i | s^n) \geq \frac{1}{n} \sum_{i=1}^n v^n_i(\hat{z}_i | s^n) - \varepsilon, \tag{47}$$

where z_i is the marginal distribution of λ regarding agent i's consumption bundle x_i and $\hat{z}_i$ is that of $\hat{\lambda}$. A mechanism $\{\varphi^n\}$ is *asymptotically surplus maximizing* if for all $\varepsilon > 0$, there exists N such that $n \geq N$ implies that $\varphi^n(\tilde{s}^n)$ is ε-surplus maximizing with a probability more than $1 - \varepsilon$.

In Section 6.2, in contrast, we consider the situation in which monetary transfers cannot be used. In this case, it is natural to consider the non-utilitarian Pareto efficiency as below. A profile of lotteries $z^{1,n} \in \Delta(X)^n$ *dominates* another profile of lotteries $z^{2,n} \in \Delta(X)^n$ at $s^n \in S^n$ if

$$v_i^n(z_i^1|s^n) \geq v_i^n(z_i^2|s^n) \quad \text{for all } i \in N_n \text{, and} \tag{48}$$

$$v_i^n(z_i^1|s^n) > v_i^n(z_i^2|s^n) \quad \text{for some } i \in N_n \tag{49}$$

A feasible random assignment $\lambda \in \Delta(X^n)$ is *ε-efficient* at $s^n \in S^n$ if either

(i) $\max_{x \in X} v_i^n(x|s^n) - v_i^n(z_i|s^n) < \varepsilon$ for all $i \in N_i$, or

(ii) there is no random assignment $\hat{\lambda} \in \Delta(X^n)$ such that $\sum_{i=1}^n \mathbb{E}_X[\hat{z}_i] \leq (1-\varepsilon)nq$ and $\hat{z}^n$ dominates z^n at s^n,

where z_i and $\hat{z}_i$ are the i-th marginal distributions of λ and $\hat{\lambda}$, respectively. A mechanism $\{\varphi^n\}$ is *asymptotically efficient* if for all $\varepsilon > 0$, there exists N such that $n \geq N$ implies that $\varphi^n(\tilde{s}^n)$ is ε-efficient with a probability more than $1-\varepsilon$.

6.2 Combinatorial Auctions

In this section, we consider a solution concept, Walrasian equilibrium, in the context of auctions that allocate (potentially) multi-unit objects. We first show the existence of the equilibrium, together with its desirable properties, efficiency, and envy-freeness. We then apply Theorem 3 to the equilibrium and show that the mechanism that approximates Walrasian equilibrium is asymptotically efficient and envy-free.

A *Walrasian equilibrium* (x^W, p^W) for economy $\lambda_V \in \Delta(V)$ consists of a measurable mapping $x^W : V \to X$ and a vector $p^W = (p_1^W, \ldots, p_L^W) \in [0,\infty)^L$ such that

(i) $\int x_\ell^W(v) \mathrm{d}\lambda(v) \leq q_\ell$ for all $\ell \in \{1, \ldots, L\}$, and $p_\ell^W = 0$ if the inequality is strict;

(ii) $x^W(v) \in D(p^W; v)$, where $D(p;v)$ is the demand correspondence defined by

$$D(p;v) = \arg\max_{x \in X} \{v(x) - p \cdot x\}.$$

Our definition of Walrasian equilibria is an extension of Gul and Stacchetti's (1999) definition and a special case of Azevedo et al.'s (2012). Gul and Stacchetti define Walrasian equilibria in finite economies where goods are gross substitutes for all agents. On the other hand, Azevedo et al. study economies with a continuum of agents and define Walrasian equilibria in lottery markets as in Hylland and Zeckhauser (1979).

The results of Azevedo et al. (2012) apply to our definition of Walrasian equilibria except for its existence, but an equilibrium does exist when indifference does not occur almost everywhere in the space of preferences V. Let $\Delta^{\mathrm{NI}}(V)$ be the set of $\lambda_V \in \Delta(V)$ such that no indifference occurs almost everywhere, i.e., $\lambda_V\{v \in V : v(x) - v(x') = c\} = 0$ for all $x, x' \in X$ and $c \in \mathbb{R}$.

Proposition 1. *A Walrasian equilibrium exists for any economy $\lambda_V \in \Delta^{\mathrm{NI}}(V)$. Any such Walrasian equilibrium is feasible, incentive compatible, surplus-maximizing, and envy-free.*

Proof. The existence is shown in Appendix. The surplus-maximization is shown by Azevedo et al. (2012). The rest of this proposition immediately follows from the definition of Walrasian equilibria. □

Proposition 1 enables us to apply Theorem 3 to Walrasian equilibria: There exist a proxy rule $P = \{P^n\}$ and a feasible, ex post incentive compatible-mechanism $\mathrm{AW} = \{\mathrm{AW}^n\}$ such that AW together with P approximates Walrasian equilibria in the sense of Theorem 3.

Theorem 4. *There exists a mechanism that is feasible, ex post incentive-compatible, asymptotically surplus maximizing, and asymptotically envy-free.*

6.3 Multi-Unit Assignment without Money

We then apply Theorem 3 to (potentially) multi-unit assignment problems without monetary transfers. In some environments, such as school choice and course allocation, the use of monetary transfers is considered undesirable (and sometimes illegal). Without such transfers, it is usually impossible to achieve a surplus-maximizing outcome, even approximately. What can be achieved in principle is non-utilitarian Pareto efficiency: Hylland and Zeckhauser (1979) define a mechanism, which we call the *HZ mechanism*, that achieves efficiency and envy-freeness.

As shown in Hylland and Zeckhauser (1979), the HZ mechanism is not incentive compatible in finite economies but is incentive compatible in the continuous limit. By Theorem 3, we can construct a mechanism that is feasible and ex post incentive compatible and whose outcomes are close to those in the HZ mechanism. Furthermore, we show that the HZ mechanism can be extended to multi-unit demands and apply Theorem 3 to the extended mechanism.[13]

We first generalize the HZ mechanism for an economy with a continuum of agents. Define the *demand correspondence* $Z(p; v)$ as the set of solutions of the following utility maximization problem:

$$Z(p; v) = \operatorname*{arg\,max}_{z \in \Delta(X)} v(z) \quad \text{subject to} \sum_{x \in X} p_x z(x) \leq 1, \tag{50}$$

[13]More precisely, we can define the extended HZ mechanism only in the limit economy, not in finite economies, but Theorem 3 guarantees the existence of a finite-economy mechanism that approximates the extended mechanism.

where $p_x = p \cdot x$. In this problem, an agent maximizes her utility function $v(z)$ subject to the constraint that her payment must not exceed the budget, i.e., 1. The agent can purchase probability shares of bundle x at price $\sum_\ell x_\ell p_\ell$, the price identical to the total price when the agent separately buys each good ℓ at price p_ℓ. For each $z \in \Delta(X)$, let $\mathbb{E}[z]$ denote $\sum_{x \in X} x \cdot z(x)$, the expected value of a random variable whose distribution is z. We define a *feasible plan* as a measurable mapping $z : V \to \Delta(X)$ such that $\int \mathbb{E}_X[z(v)]\mathrm{d}\lambda(v) \leq q$.

Definition 1. A *generalized Hylland-Zeckhauser equilibrium* (or simply an *HZ equilibrium*) for economy $\lambda \in \Delta(V)$ is a pair $(z^{\mathrm{HZ}}, p^{\mathrm{HZ}})$ that consists of a feasible plan $z^{\mathrm{HZ}} : V \to \Delta(X)$ and price vector $p^{\mathrm{HZ}} = (p_1^{\mathrm{HZ}}, \ldots, p_L^{\mathrm{HZ}}) \in [0, \infty)^L$ such that

(i) $z^{\mathrm{HZ}}(v) \in Z(p^{\mathrm{HZ}}; v)$; and

(ii) $\int \mathbb{E}_X[z^{\mathrm{HZ}}(v)]\mathrm{d}\lambda(v) \leq q$ and $p_\ell^{\mathrm{HZ}} = 0$ if the inequality is strict at the ℓ-th dimension.[14]

We define the *generalized HZ mechanism* as a ∞-mechanism whose outcome is selected from HZ equilibrium outcomes. Such a mechanism exists because Proposition 2 ensures that any distribution on V has at least one HZ equilibrium.

Proposition 2. *A generalized Hylland-Zeckhauser equilibrium exists for any $\lambda \in \Delta(V)$.*

Proof. See Appendix A.7. □

This generalized solution concept inherits two desirable properties, efficiency and envy-freeness, which the original HZ equilibrium possesses. We say that a feasible plan z is *efficient* at $\lambda_V \in \Delta(V)$ if $\lambda_V\{v : v(z'(v)) > v(z(v))\} > 0$ implies $\lambda_V\{v : v(z'(v)) < v(z(v))\} > 0$ for all feasible plans z'. A feasible plan z is *envy-free* if $v(z(v)) \geq v(z(\hat{v}))$ for all $v, \hat{v} \in V$.

Proposition 3. *Let $(z^{\mathrm{HZ}}, p^{\mathrm{HZ}})$ be an HZ equilibrium for economy $\lambda_V \in \Delta(V)$. Then,*

(i) *z^{HZ} is envy-free;*

(ii) *z^{HZ} is efficient at λ_V if $\lambda_V\{v : \#\arg\max_{x \in X} v(x) = 1\} = 1$.*[15]

Proof. See Appendix A.8. □

In our environment, the assumption of the second statement is always satisfied. This fact becomes a special case of the following result, corresponding to the case $B = X \times \{0\}$.

Lemma 1. *For all $B \in \mathcal{B}_f$ and $\theta \in \Theta$, the set*

$$I(B, \theta) = \left\{ s \in S : u^\infty(y|s, \theta) = u^\infty(y'|s, \theta) \text{ for some } y, y' \in B,\ y \neq y' \right\} \tag{51}$$

has Lebesgue measure 0.

[14] As in Budish et al. (2012), we need the latter half of the condition (ii) to obtain the efficiency of HZ equilibria.

[15] This requirement on utility functions also appears in Budish et al. (2012).

Proof. See Appendix. □

The above lemma is also useful to establish the fact that an HZ equilibrium is indeed implementable by a finite budget set. Recall that in Theorem 3 the budget rule B^∞ was assumed to be finite. Even though a budget set in the HZ market consists of a continuum of choices, it possesses only finitely many extreme points as a compact set represented by a linear inequality system:

$$\sum_{x \in X} z(x) = 1 \tag{52}$$

$$\sum_{x \in X} p_x z(x) \leq 1 \quad (p_x = p \cdot x) \tag{53}$$

$$0 \leq z(x) \leq 1 \text{ for all } x \in X. \tag{54}$$

Therefore, by Lemma 1, all the agents except for those in a measure-0 set have no indifference between extreme points and thus never choose a non-extreme point.

From these observations, we can apply Theorem 3 to the infinite-economy version of the HZ mechanism.

Theorem 5. *There exists a money-free mechanism that is feasible, ex post incentive compatible, asymptotically efficient, and asymptotically envy-free.*

Proof. See Appendix A.5. □

7 Discussion

In the present paper, we developed a technique for approximating continuous-market mechanisms by finite-market mechanisms. We generalized the random priority mechanism for our approximation method, showing that the generalization, which we called the generalized random priority mechanism with budgets (GRP), is always feasible and ex post incentive compatible. Further, we have established that a broad class of continuous-market mechanisms can be approximated by GRP with appropriately designed budget sets whenever they satisfy the requirements for feasibility and incentive compatibility in addition to a mild finiteness condition. We also applied our approximation technique to multi-unit auctions and allocation problems without money.

Our approximation results confirm a direct connection between continuous- and finite-market phenomena in the context of mechanism design. Continuous-market results are often considered to be "benchmarks" that may or may not be approximately applicable to finite markets. Such continuous-market results are indeed more than benchmarks according to our findings: Something achievable in continuous markets is also approximately achievable in large finite markets.

A Proofs

This appendix provides proofs not presented in the main body of the paper. Before the proofs, we introduce several notations for convenience. When a measure λ depends on some variable α, we write $\lambda(A|\alpha)$ to denote $\lambda(\alpha)(A)$. Define

$$\mu(T|\theta) = \int_T f(s|\theta)\mathrm{d}s \quad \text{for measurable } T \subseteq S. \tag{55}$$

Let $\mu_V(\theta)$ be the image measure of $\mu(\theta)$ with respect to the mapping $s_i \mapsto v^\infty(\cdot|s_i, \theta)$:

$$\mu_V(W|\theta) = \mu(\{s \in S : v^\infty(\cdot|s, \theta) \in W\}|\theta) \tag{56}$$

for measurable $W \subseteq V$. Let $\mu_V^n(\theta)$ be the corresponding product measure:

$$\mu_V^n(W_1 \times \cdots \times W_n|\theta) = \mu_V(W_1) \cdots \mu_V(W_n).$$

A.1 Proof of Lemma 1

It suffices to show that $J = \{s \in S : u^\infty(z|s, \theta) = u^\infty(z'|s, \theta)\}$ has measure 0 in $\mu(\theta)$, for all $z, z' \in B$ such that $z \neq z'$. Define $g(s) = u^\infty(z|s, \theta) - u^\infty(z'|s, \theta)$. Thanks to Assumption 1, $g^{-1}(0)$ is either the empty set or a $(d(S)-1)$-dimensional manifold; in either case its Lebesgue measure in the $d(S)$-dimensional space S is 0. Therefore, $\mu(J|\theta) = \mu(g^{-1}(0)|\theta) = 0$.

A.2 Utility Convergence Lemma

We prove a lemma that will be used in subsequent proofs. The lemma asserts a statistical fact that finite-economy utility functions converge to the corresponding infinite economy ones as n goes to infinity, even if one single signal can be erroneous but believed to be correct.

Definition 2. $\omega^R \in \Omega$ is *ε-regular (in preferences)* at n if

$$\left|v_i^n(x|s_j', s_{-j}^{R,n}) - v^\infty(x|s_i^R, \theta^R)\right| < \varepsilon$$

for all $x \in X$, $i, j \in N_n$, and $s_j' \in S$, where $\theta^R = \tilde{\theta}(\omega^R)$ and $s_k^R = \tilde{s}_k(\omega^R)$.

Lemma 2. *For all $\varepsilon > 0$, the inner probability measure of the set $\{\omega^R \in \Omega : \omega^R$ is ε-regular at $n\}$ converges to 1 as $n \to \infty$.*

Proof of Lemma 2. We define *ε-regularity in beliefs* and establish that the following statement is sufficient to prove this lemma:

$$\begin{aligned}&\textit{The inner probability of } \left\{\omega^R \in \Omega : \omega^R \textit{ is } \varepsilon\textit{-regular in beliefs at } n\right\}\\ &\textit{converges to 1 as } n \to \infty.\end{aligned} \tag{57}$$

Let $P^\varepsilon(\theta, s^n) = \mathbb{P}\{\|\tilde{\theta} - \theta\| < \varepsilon|\tilde{s}^n = s^n\}$ be the conditional probability that the true state is ε close to θ given the observed signal profile is (believed to be) s^n.

Definition 3. $\omega^R \in \Omega$ is a *ε-regular in beliefs* at n if

$$P^{\varepsilon}\big(\theta^R, (\hat{s}_j, s_{-j}^{R,n})\big) > 1 - \varepsilon$$

with $\theta^R = \tilde{\theta}(\omega^R)$ and $s_{-j}^{R,n} = \tilde{s}_{-j}^n(\omega^R)$, for all $j \in \{1, \ldots, n\}$ and $\hat{s}_j \in S$.

That is, agent i assigns a probability more than $1 - \varepsilon$ to the ε-neighborhood of the true state θ^R even when some agent j's report, which agent i believes is correct, is erroneous.[16]

Lemma 3 establish the link between the two regularity concepts.

Lemma 3. *For all $\varepsilon_p > 0$, there exists $\varepsilon_b > 0$ such that $\omega^R \in \Omega$ is ε_b-regular in beliefs only if it is ε_p-regular in preferences.*

Proof. Since v^{∞} is uniformly continuous, for any $\varepsilon_p > 0$, there exists $\varepsilon_b \in (0, \varepsilon_p/3)$ such that $\|\theta - \theta'\| < \varepsilon_b$ implies $\|v^{\infty}(x|s_i, \theta) - v^{\infty}(x|s_i, \theta')\| < \varepsilon_p$ for all $x \in X$ and $s_i \in S$. Suppose that ω^R is ε_b-regular in beliefs. Then, for all $x \in X$ and $\hat{s}_j \in S$,

$$\begin{aligned}
&\big|v_i^n(x|\hat{s}_j, s_{-j}^{R,n}) - v^{\infty}(x|s_i^R, \theta^R)\big| \\
&\quad = \left|\int \big[v^{\infty}(x|s_i^R, \theta) - v^{\infty}\big(x|s_i^R, \theta^R\big)\big]\, f(\theta|\hat{s}_j, s_{-j}^{R,n})\, \mathrm{d}\theta\right| \\
&\quad \leq \int \big|v^{\infty}(x|s_i^R, \theta) - v^{\infty}(x\big|s_i^R, \theta^R)\big|\, f(\theta|\hat{s}_j, s_{-j}^{R,n})\, \mathrm{d}\theta \\
&\quad \leq \int_{|\theta - \theta^R| < \varepsilon_b} \varepsilon_p\, f(\theta|\hat{s}_j, s_{-j}^{R,n})\, \mathrm{d}\theta + \int_{|\theta - \theta^R| \geq \varepsilon_b} 2\, f(\theta|\hat{s}_j, s_{-j}^{R,n})\, \mathrm{d}\theta \\
&\quad \leq \varepsilon_p + 2\varepsilon_b < \varepsilon_p.
\end{aligned}$$

That is, ω^R is ε_p-regular in preferences. ☐

Thus it suffices to show condition (57). Its proof consists of five steps: Steps 1 and 2 are used in Step 3, and Steps 3 and 4 are used in Step 5.

Step 1: The inverse of $\mu^n(\theta)$ is uniformly continuous. Since μ^n is a continuous injection from the compact space Θ to the Polish space $\Delta(S)$, its inverse $(\mu^n)^{-1}$ is a continuous mapping from $\mu^n(\Theta)$ to Θ (see, e.g., Munkres, 2000, Theorem 26.6). Furthermore, $(\mu^n)^{-1}$ is uniformly continuous because its domain $\mu(\Theta)$ is compact.

[16] Readers should carefully distinguish $\tilde{\theta}$, the true state for the *agents*, from θ^R, the state that *we*, as modelers, know is the true state. The agents never know that θ^R is the true state, but they indeed assign a high probability on the neighborhood of θ^R without knowing what is θ^R.

Step 2: For all $\alpha, \beta > 0$, $\mathbb{P}\{Q^\alpha(\tilde{s}^n) \geq 1-\beta\}$ converges to 1 as n goes to ∞. Here, $Q_\alpha(s^n)$ is the conditional probability that the true signal distribution $\mu(\theta)$ and the empirical distribution $\mu^{\mathrm{emp}}(s^n) = \sum_{i=1}^n \delta_{s_i}/n$ (δ_{s_i} is the Dirac measure) are ε-close, given that the realized signal profile is s^n.[17] The distance is measured by the Lévy–Prokhorov metric d_{LP}, which is consistent with the weak convergence of measures. Formally, $Q^\alpha(s^n)$ is defined by

$$Q^\alpha(s^n) = \mathbb{P}\big(d_{\mathrm{LP}}(\mu(\tilde{\theta}), \mu^{\mathrm{emp}}(s^n)) < \alpha \big| \tilde{s}^n = s^n\big).$$

This step is shown by the following calculations:

$$\begin{aligned} 1 - \mathbb{P}\big\{Q^\alpha(\tilde{s}^n) \geq 1-\beta\big\} &= \beta^{-1} \cdot \beta \cdot \mathbb{P}\big\{1 - Q^\alpha(\tilde{s}^n) > \beta\big\} \\ &\leq \beta^{-1} \cdot \mathbb{E}[1 - Q^\alpha(\tilde{s}^n)] \\ &= \beta^{-1} \cdot \mathbb{P}\big\{d_{\mathrm{LP}}(\mu(\tilde{\theta}), \mu^{\mathrm{emp}}(\tilde{s}^n)) \geq \alpha\big\}. \end{aligned}$$

The last term converges to 0 as $n \to \infty$ because empirical distributions weakly converge to the true distribution by the Glivenko–Cantelli theorem.

Step 3: For all $\alpha, \beta > 0$, $\mathbb{P}\{P^\alpha(\tilde{\theta}, \tilde{s}^n) \geq 1-\beta\}$ converges to 1 as n goes to ∞. Fix $\varepsilon^* > 0$. By Step 1, there exists $\delta > 0$ such that $d_{\mathrm{LP}}(\mu(\theta), \mu(\theta')) \leq 2\delta$ implies $\|\theta - \theta'\| \leq \alpha$ for all $\theta, \theta' \in \Theta$. By Step 2, there exists N such that $n \geq N$ implies that the event $E^n = \{Q^\delta(\tilde{s}^n) \geq 1-\beta$ and $d_{\mathrm{LP}}(\mu(\tilde{\theta}), \mu^{\mathrm{emp}}(\tilde{s}^n)) < \delta\}$ has a probability more than $1 - \varepsilon^*$. Let $\omega^R \in E^n$, $\theta^R = \tilde{\theta}(\omega^R)$ and $s^{R,n} = \tilde{s}^n(\omega^R)$. By the above arguments, we obtain

$$\begin{aligned} P^\alpha(\theta^R, s^{R,n}) &= \mathbb{P}\left(\|\tilde{\theta} - \theta^R\| \leq \alpha \middle| \tilde{s}^n = s^{R,n}\right) \\ &\geq \mathbb{P}\left(d_{\mathrm{LP}}\big(\mu^n(\tilde{\theta}), \mu^n(\theta^R)\big) \leq 2\delta \middle| \tilde{s}^n = s^{R,n}\right) \\ &\geq \mathbb{P}\left(d_{\mathrm{LP}}(\mu(\tilde{\theta}), \mu^{\mathrm{emp}}(s^{R,n})) \leq \delta \middle| \tilde{s}^n = s^{R,n}\right) \\ &= \mathbb{P}\left(d_{\mathrm{LP}}(\mu(\tilde{\theta}), \mu^{\mathrm{emp}}(\tilde{s}^n)) \leq \delta \middle| \tilde{s}^n = s^{R,n}\right) \\ &\geq Q^\delta(s^{R,n}) \geq 1 - \beta. \end{aligned}$$

Step 4: $1 - P^\varepsilon(\theta, (\hat{s}_j, s^n_{-j})) \leq (1 - P^\varepsilon(\theta, \hat{s}^n))M/m$ for all $n \in \mathbb{N}$, $j \in N_n$, $\theta \in \Theta$, $s^n \in S^n$, and $\hat{s}_j \in S$. Here, M and m are the maximum and minimum of the ratio $f(s|\theta)/f(s'|\theta')$ over $s, s' \in S$ and $\theta, \theta' \in \Theta$. The probability $1 - P^\varepsilon(\theta, (\hat{s}_j, s^n_{-j}))$ is evaluated from above by Bayes' law:

$$1 - P^\varepsilon(\theta, (\hat{s}_j, s^n_{-j})) \leq \frac{AM}{AM + Bm} \leq \frac{A}{B} \cdot \frac{M}{m} \leq (1 - P^\varepsilon(s^n)) \cdot \frac{M}{m},$$

[17] A Dirac measure δ_x is a probability measure such that $\delta_x(\{x\}) = 1$.

where

$$A = \int_{\|\theta-\theta'\|\geq\varepsilon} f(\theta)f(s_1|\theta)\cdots f(s_n|\theta)\mathrm{d}\theta'$$
$$B = \int_{\|\theta-\theta'\|<\varepsilon} f(\theta)f(s_1|\theta)\cdots f(s_n|\theta)\mathrm{d}\theta'.$$

Step 5. Let $\varepsilon > 0$. By Step 4, $P^\varepsilon(\tilde{\theta}, \tilde{s}^n) \geq 1 - (m/M)\varepsilon$ implies

$$P^\varepsilon(\tilde{\theta}, (\hat{s}_j, \tilde{s}^n)) \geq 1 - (1 - P^\varepsilon(\tilde{\theta}, \tilde{s}^n)) \cdot \frac{M}{m} \geq 1 - \varepsilon.$$

Hence, $\mathbb{P}\{P^\varepsilon(\tilde{\theta}, \tilde{s}^n) \geq 1-(m/M)\varepsilon\}$ is at most as high as $\mathbb{P}_*\{P^\varepsilon(\tilde{\theta}, (\hat{s}_j, \tilde{s}^n)) \geq 1-\varepsilon\}$, the inner probability of ε-regularity in beliefs. From Step 3, we know that $\mathbb{P}\{P^\varepsilon(\tilde{\theta}, \tilde{s}^n) \geq 1-(m/M)\varepsilon\}$ converges to 1 as n goes to ∞ and therefore that the inner probability of ε-regularity in beliefs also converges to 1; this is the condition (57). □

A.3 Proof of Theorems 2

In this proof, we use Lemma 1 (Sections 6.3 and A.1) and Lemma 2 (Section A.2). Let $P = \mathbb{P}\{\tilde{\theta} \in \Theta_\mathrm{C}\}$. Take arbitrary $\varepsilon > 0$ and let

$$\varepsilon' = \frac{P\min\{1, q_*\}}{2\bar{x}(|X|+10)} \cdot \varepsilon \tag{58}$$

where $q_* = \min\{q_1, \ldots, q_L\}$.

Step 1: *Application of Lemma 2.*

By Lemma 2, there exists N_1 such that $n \geq N_1$ implies that $\mathbb{P}_*(E_1^n) > 1 - \varepsilon'$, where

$$E_1^n = \left\{\left|u_i^n(x|\tilde{s}^n) - u^\infty(x|\tilde{s}_i, \tilde{\theta})\right| < \varepsilon' \text{ for all } i \in N_n \text{ and } x \in X\right\}. \tag{59}$$

Step 2: *Optimal choices and their robustness.*

We first introduce the *demand correspondences* for finite and infinite economies:

$$Y_i^n(B|s^n) = \arg\max_{(z,t)\in B} u_i^n(z, t|s^n) \tag{60}$$
$$Y^\infty(B|s, \theta) = \arg\max_{(z,t)\in B} u^\infty(z, t|s, \theta) \tag{61}$$

for each $B \in \mathcal{B}_f$, $s^n \in S^n$, $s \in S$, and $\theta \in \Theta$. In addition, define the following "buffered" demand correspondence:

$$Y_i^n(\varepsilon_B, B|s^n) = \bigcup\nolimits_{d_H(B,B')\leq\varepsilon_B,\, i\neq j,\, \hat{s}_j\in S} Y_i^n(B'|\hat{s}_j, s_{-j}^n), \tag{62}$$

where d_H is the Hausdorff metric, which metricize the continuity of correspondences; in particular, $B : \Theta \to \mathcal{B}_f$ is both upper- and lower-hemicontinuous at θ if and only if $\lim_{\theta' \to \theta} d_H(B(\theta), B(\theta')) = 0$ (see, e.g., Ok, 2007).

The next lemma claims that the fraction M^n of the agents whose finite-economy and infinite-economy choices are δ-close is more than $1 - \varepsilon$ with probability 1, as $n \to \infty$. Its proof is shown later (Section A.3.1). For its formal statement, we endow $\Delta(X) \times \mathbb{R}$ with the following L^∞-norm:

$$\|(z, t) - (z', t')\|_\infty = \max(\{|z(x) - z'(x)| : x \in X\} \cup \{|t - t'|\}). \tag{63}$$

Lemma 4. *Suppose that $(\varphi^\infty, B^\infty)$ is an incentive compatible ∞-mechanism. For all $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that*

$$\lim_{n \to \infty} \mathbb{P}_* \left\{ M^n(\delta; \tilde{s}^n, \tilde{\theta}) \geq 1 - \varepsilon \right\} = 1, \tag{64}$$

where

$$M^n(\delta; \tilde{s}^n, \tilde{\theta}) = \frac{1}{n} \cdot \# \Big\{ i \in N_n : Y^\infty(B^\infty(\tilde{\theta}) | \tilde{s}_i, \tilde{\theta}) = \{y_i^{(\infty)}\} \text{ and } \\ \big\| y_i^{(\infty)} - y_i^{(n)} \big\|_\infty < \delta \text{ for all } y_i^{(n)} \in Y_i^n(\delta, B^\infty(\tilde{\theta}) | \tilde{s}^n) \Big\}. \tag{65}$$

By Lemma 4, there exist $\varepsilon_B \in (0, \varepsilon')$ and $N_2 > N_1$ such that $n \geq N_2$ implies that

$$E_2^n = \left\{ M^n(\varepsilon_B; \tilde{s}^n, \tilde{\theta}) \geq 1 - \varepsilon' \right\} \tag{66}$$

has a inner probability more than $1 - \varepsilon'$. Notice that in the definition of M^n, the buffered version of Y^n is used. That is, in the definition, each agent i in finite economies may face a budget set that is slightly different from the infinite-economy budget set $B^\infty(\tilde{\theta})$ and another agent j's signal s_j can be inaccurate.

Step 3: *Maximum liklihood estimation of the infinite-economy budget rule.*

Let $\theta^{\mathrm{MLE}}(s^n)$ be the maximum likelihood estimator (MLE) and define

$$B_i^n(s_{-i}^n) = B^\infty(\theta^{\mathrm{MLE}}(s_{-i}^n)). \tag{67}$$

Note that the following regularity conditions are satisfied:

(i) The state space Θ is compact.

(ii) The conditional density $f(s_i|\theta)$ is continuous as a function of (s_i, θ).

(iii) $\max_{s_i, \theta} |\log f(s_i|\theta)|$ is finite.

(iv) For all $\theta \neq \theta'$, there exists $s_i \in S$ such that $f(s_i|\theta) \neq f(s_i|\theta')$.

We thus can apply the techniques of Amemiya (1985, Theorem 4.1.1 and Section 4.2) and obtain the following:

(i) $\mathtt{L}(\hat{\theta}|\theta) = \int_S \log f(s_i|\hat{\theta})\, f(s_i|\theta)\, \mathrm{d}s_i$ is a continuous function of $\hat{\theta}$ with a unique maximizer.

(ii) $\theta^{\mathrm{MLE}}(\tilde{s}^n)$ converges in probability to $\tilde{\theta}$ as n goes to ∞.

Since the contribution of each $\log f(s_i|\hat{\theta})$ to the mean $n^{-1}\sum_i \log f(s_i|\hat{\theta})$ is bounded by $\max_{s_i,\theta} |f(s_i|\theta)|/n$, which disappears in the limit, we have

$$\operatorname*{plim}_{n\to\infty} \max_i \left\| \theta^{\mathrm{MLE}}(\tilde{s}^n_{-i}) - \tilde{\theta} \right\| = 0.$$

By Lemma 5 below, this implies that there exists $N_3 > N_2$ such that for all $n \geq N_3$, the inner probability $\mathbb{P}_*(E_3^n)$ is greater than $P - \varepsilon'$, where

$$E_3^n = \left\{ \max_{i \in N_n} d_H(B^\infty(\tilde{\theta}), B_i^n(\tilde{s}^n_{-i})) < \varepsilon_B \right\} \cap \left\{ \tilde{\theta} \in \Theta_{\mathrm{C}} \right\}.$$

Lemma 5. *Let X be a compact subset of $\mathbb{R}^d$ and (Y, d) a compact metric space. Consider a mapping $g : X \to Y$ such that g is continuous at every $x \in X'$, where $X' \subseteq X$ is a Lebesgue measurable set with a positive Lebesgue measure. For all $\varepsilon > 0$, there exists $\delta > 0$ and a compact set $X^* \subseteq X$ such that*

(i) $m(X^*) \geq (1-\varepsilon)m(X')$ and

(ii) $d(g(x^*), g(x)) < \varepsilon$ whenever $\|x^* - x\| < \delta$ for all $x^* \in X^*$ and $x \in X$.

Proof. Since the Lebesgue measure is regular, there exists a compact set $X^* \subseteq X'$ such that $m(X^*) > (1-\varepsilon)m(X')$. For each $x \in X^*$, let

$$\delta_x = \frac{1}{2} \inf\big\{ \|x_k - x'\| : d(g(x_k), g(x')) \geq \varepsilon/2 \big\}, \tag{68}$$

which must be positive by the definition of continuity. Let $U(\delta, x) = \{x' \in X : \|x - x'\| < \delta\}$ be the δ-open neighborhood of x. Since X^* is compact, an open covering $\{U(\delta_x, x)\}_{x \in X^*}$ of X^* has a finite sub-covering $\{U(\delta_{x_k}, x_k)\}_{k=1}^K$.

Let $\delta = \min_k \delta_{x_k}$, and suppose that $x^* \in X^*$, $x \in X$, and $\|x^* - x\| < \delta$. Since $\{U(\delta_{x_k}, x_k)\}_{k=1}^K$ covers X^*, there exists k such that $\|x^* - x_k\| < \delta_{x_k}$. This implies $\|x - x_k\| \leq \|x - x^*\| + \|x^* - x_k\| < 2\delta_{x_k}$, and thus

$$d(g(x^*), g(x)) \leq d(g(x^*), g(x_k)) + d(g(x_k), g(x)) < \varepsilon$$

by (68). □

Step 4: *Approximate feasibility.*

We now concretely implement lotteries by the following method that is consistent with the last step of GRP (Step (iii)(b)): For each $z_i \in \Delta(X)$, define $Lz_i(\cdot)$ as a function from $(0,1]$ to X such that

$$Lz_i(r_i) = x^{(K)} \quad \text{if } \textstyle\sum_{k<K} z_i(x^{(k)}) < r_i \leq \sum_{k\leq K} z_i(x^{(k)}), \tag{69}$$

where $x^{(1)}, \dots, x^{(|X|)}$ are the elements of X sorted in the lexicographic order. Using the operator L, we define W_i^n and e_i^n: W_i^n is a function of randomization device r_i that serves as an upperbound of agent i's consumption and e_i^n is the expected value of W_i^n. Formally,

$$W_i^n(r_i; \varepsilon_B, B|s^n) = \sup\bigl\{Lz_i(r_i) : (z_i, t_i) \in Y_i^n(\varepsilon_B, B|s^n)\bigr\} \tag{70}$$

$$e_i^n(\varepsilon_B, B|s^n) = \int_0^1 W_i^n(r_i|\varepsilon_B, B; s^n)\mathrm{d}r_i. \tag{71}$$

The supremum is taken for each good ℓ. Recall $Lz_i(r_i) \in \mathbb{R}^L$.

We then find sufficiently large n with which allocation is approximately feasible even in the worst case. Recall that φ^∞ is α-feasible. Thus, by the weak law of large numbers, there exists $N_4 > N_3$ such that $\mathbb{P}_*(E_4^n) > 1 - \varepsilon'$ for all $n \geq N_4$, where

$$\begin{aligned} E_4^n = &\left\{\frac{1}{n}\sum_{i=1}^n \mathbb{E}_X\bigl[z^\infty(\tilde{s}_i|\tilde{\theta})\bigr] \leq (1+\alpha+\varepsilon')q\right\} \\ &\cap \left\{\left|\frac{1}{n}\sum_{i=1}^n t^\infty(\tilde{s}_i|\tilde{\theta}) - \int t^\infty(s|\tilde{\theta})f(s|\tilde{\theta})\mathrm{d}s\right| < \varepsilon'\right\} \end{aligned} \tag{72}$$

and $(z^\infty, t^\infty) = \varphi^\infty$. Since $\varphi^\infty(\tilde{s}_i|\tilde{\theta}) = Y^\infty(B^\infty(\tilde{\theta})|\tilde{s}_i, \tilde{\theta})$ almost surely by Lemma 1, the following holds almost surely as well conditional on $E^n = E_1^n \cap E_2^n \cap E_3^n \cap E_4^n$:

$$\frac{1}{n}\sum_{i=1}^n e_i^n(\varepsilon_B, B^\infty(\tilde{\theta})|\tilde{s}^n) \leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}_X\bigl[z^\infty(\tilde{s}_i|\tilde{\theta})\bigr] + 2\varepsilon_B|X|^2\bar{\mathbf{x}} \tag{73}$$

$$\leq (1+\alpha+3\varepsilon')q, \tag{74}$$

where $\bar{\mathbf{x}}$ is a L-dimensional vector whose elements are all $\bar{x}$. In the first inequality, we used the fact that

$$\begin{aligned} e_i^n - \mathbb{E}_X\bigl[z^\infty(\tilde{s}_i|\tilde{\theta})\bigr] &\leq \mathfrak{m}\left\{r_i : W_i^n(r_i) \neq L[z^\infty(\tilde{s}_i|\tilde{\theta})](r_i)\right\}\bar{\mathbf{x}} \\ &\leq \sum_{x\in X} 2\varepsilon_B|X|\bar{\mathbf{x}} \leq 2\varepsilon_B|X|^2\bar{\mathbf{x}}. \end{aligned}$$

Step 5: *Availability of optimal choices.*

In this step, we show that agents do not suffer from the feasibility constraint with probability approximately equal to 1 for sufficiently large n.

Let $\mathcal{P}^n$ be the set of priorities (strict total orders) on N_n, and let $\mathcal{P}(i,m)$ be the set of the priorities in which agent i has the m-th priority. We consider the case of $m \leq M_n \equiv n/(1+\alpha+4\varepsilon')$. Let $\mathbb{Q}^n$ be the probability measure that represents the probabilistic structure on $(\succ, r_1, \ldots, r_n)$; i.e., $\mathbb{Q}^n$ is the probability measure on $\mathcal{P} \times [0,1]^n$ uniquely determined by the following independence condition:

$$\mathbb{Q}^n(P \times T_1 \times \cdots \times T_n) = \frac{\#P}{n!} \cdot \mathfrak{m}(T_1) \cdots \mathfrak{m}(T_n) \tag{75}$$

for all $P \subseteq \mathcal{P}$ and Lebesgue measurable subsets $T_1, \ldots, T_n \subseteq [0,1]$. Let $A_{i,m}$ denote the event $\{\succ \in \mathcal{P}(i,m)\}$.

In this step, we consider the probability space induced by $\mathbb{Q}^n$ in order to prove the following statement with fixed $\omega \in E_1^n \cap E_3^n$: There exists $N_4 > N_{3,2}$ such that for all i and $n \geq N_4$, the inner probability of

$$\bar{\mathbf{x}} + \sum_{j \succ i} \max_{s_i' \in S} x_j(\succ, r^n; s_i', \tilde{s}_{-i}^n) \geq nq \tag{76}$$

conditional on $(\bigcup_{m \leq M_n} A_{i,m})$ is more than $1 - \varepsilon$. Recall that the measure is $\mathbb{Q}^n$ and ω is fixed so that $\succ$ and r_i are random variables while $\tilde{s}_i$ and $\tilde{\theta}$ are deterministic. To show (76), we instead consider the following condition:

$$\bar{\mathbf{x}} + \sum_{j \succ i} W_j^n(r_j; \varepsilon_B, B^\infty(\tilde{\theta}) | \tilde{s}^n) \geq nq. \tag{77}$$

This condition implies (76) because

$$\max_{s_i' \in S} x_j(\succ, r^n; s_i', \tilde{s}_{-i}^n) \leq W_j^n(r_j; \varepsilon_B, B^\infty(\tilde{\theta}) | \tilde{s}^n)$$

as long as the condition (76) is satisfied for all $i \succ j$. We simply write $W_j^n(r_j)$ to denote $W_j^n(r_j; \varepsilon_B, B^\infty(\tilde{\theta}) | \tilde{s}^n)$.

To show (77), we apply Chebyshev's inequality to the summation $\sum_{j \succ i} W_j^n(r_j) = \sum_{j \neq i} h_j W_j^n(r_j)$, where $h_j = 1\{j \succ i\}$. To this end, we evaluate the conditional mean and variance of $W_j^n(r_j)$ as below. First, we evaluate an upper bound of the conditional mean from above:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^n}\left[\sum_{j \succ i} W_j^n(r_j) \middle| A_{i,m}\right] &= \sum_{j \neq i} \mathbb{E}^{\mathbb{Q}^n}\left[h_j \middle| A_{i,m}\right] \cdot e_j^n & (78)\\
&= \frac{m-1}{n-1} \sum_{j \neq i} e_j^n & (79)\\
&\leq \frac{m}{n}(1+\alpha+3\varepsilon')nq & (80)\\
&\leq \frac{1+\alpha+3\varepsilon'}{1+\alpha+4\varepsilon'} nq & (81)
\end{aligned}$$

by the independence of $(\succ, r_1, \ldots, r_n)$ and then (74). Second, the conditional variance in each dimension ℓ is bounded above by $n\bar{x}^2$:

$$
\begin{aligned}
\mathrm{Var}^{\mathbb{Q}^n}\left[\sum_{j\succ i} W^n_{j,\ell}(r_j)\middle| A_{i,m}\right] &= \sum_{j,k\in N_n\setminus\{i\}} \mathrm{Cov}^{\mathbb{Q}^n}\left(h_j e^n_{j,\ell}, h_k e^n_{k,\ell}\middle| A_{i,m}\right) && (82)\\
&\le n\bar{x}^2 + \sum_{j,k\in N_n\setminus\{i\},\, j\neq k} \mathrm{Cov}^{\mathbb{Q}^n}\left(h_j e^n_{j,\ell}, h_k e^n_{k,\ell}\middle| A_{i,m}\right) && (83)\\
&\le n\bar{x}^2. && (84)
\end{aligned}
$$

To derive the last inequality, we use the fact $\mathrm{Cov}^{\mathbb{Q}^n}(h_j e^n_{j,\ell}, h_k e^n_{k,\ell}\big| A_{i,m}) \le 0$.

The covariance is calculated as follows. (To make notations simple, we omit the symbols $\mathbb{Q}^n$ and $A_{i,m}$ from the expectation and covariance operators, but in this paragraph we keep considering the probability space $\mathbb{Q}^n$ conditional on $A_{i,m}$.)

$$
\begin{aligned}
\mathrm{Cov}(h_j e^n_{j,\ell}, h_k e^n_{k,\ell}) &= \mathbb{E}\big[\big(\varDelta h_j e^n_{j,\ell} + \varDelta e^n_{j,\ell}\mathbb{E}[h_j]\big)\big(\varDelta h_k e^n_{k,\ell} + \varDelta e^n_{k,\ell}\mathbb{E}[h_k]\big)\big] && (85)\\
&= \mathrm{Cov}(h_j, h_k)\mathbb{E}[e^n_{j,\ell}]\mathbb{E}[e^n_{j,\ell}], && (86)
\end{aligned}
$$

where $\varDelta h_j = h_j - \mathbb{E}[h_j]$ and $\varDelta e^n_{j,\ell} = e^n_{j,\ell} - \mathbb{E}[e^n_{j,\ell}]$. Further, $\mathrm{Cov}(h_j, h_k)$ is negative because

$$
\begin{aligned}
\mathrm{Cov}(h_j, h_k) = &\frac{(m-1)(m-2)}{(n-1)(n-2)} \cdot \left(1 - \frac{m-1}{n-1}\right)^2 \\
&+ \frac{(n-m)(n-m-1)}{(n-1)(n-2)} \cdot \left(\frac{m-1}{n-1}\right)^2 \\
&- \frac{2(m-1)(n-m)}{(n-1)(n-2)} \cdot \left(\frac{m-1}{n-1}\right)\left(1 - \frac{m-1}{n-1}\right) && (87)\\
= &\frac{1}{(n-1)^3(n-2)} \cdot \Big[(m-1)(m-2)\cdot(n-m)^2 \\
&+ (n-m)(n-m-1)\cdot(m-1)^2 \\
&- 2(m-1)(n-m)\cdot(m-1)(n-m)\Big] && (88)\\
= &-\frac{(m-1)(n-m)}{(n-1)^2(n-2)} \le 0, && (89)
\end{aligned}
$$

From the above facts, there exists $N_4 > \max\{N_{3,2}, 1/\varepsilon'\}$ such that for all $n \ge N_4$, $\omega \in E^n_1 \cap E^n_3$, $i \in N_n$, and $m \le M_n$, there exists a measurable set $C^n_{i,m}(\omega) \subseteq \mathcal{P}(i,m) \times [0,1]^n$ such that

$$\mathbb{Q}^n(C^n_{i,m}(\omega)|A_{i,m}) > 1 - \varepsilon' \tag{90}$$

and (77) holds whenever $(\succ, r^n) \in C^n_{i,m}(\omega)$. Let

$$C^n_i(\omega) = \bigcup_{m\le M_n} C^n_{i,m}(\omega). \tag{91}$$

Since $\mathbb{Q}^n(A_{i,m}) = 1/n$ for each m,

$$\mathbb{Q}^n(C_i^n(\omega)) > (1-\varepsilon') \cdot \frac{1-1/n}{1+\alpha+4\varepsilon'} > 1-(\alpha+6\varepsilon'). \tag{92}$$

Step 6: *Approximation inequalities.*

We consider $E^n = E_1^n \cap \cdots \cap E_4^n$, whose conditional inner probability $\mathbb{P}_*(E^n|\tilde{\theta} \in \Theta_{\mathrm{C}})$ is more than $1-\varepsilon$. Fix $\omega \in E^n$. We first evaluate agent i's utility level from below:

$$\begin{aligned} u_i^n(\mathrm{GRP}_i(\tilde{s}^n)|\tilde{s}^n) &\geq u_i^n(Y_i^n(B_i^n(\tilde{s}_{-i}^n)|\tilde{s}^n)|\tilde{s}^n) - (\alpha+6\varepsilon') && \text{(by Step 5)} \\ &\geq u_i^n(\varphi^\infty(\tilde{s}_i|\tilde{\theta})|\tilde{s}^n) - [\alpha+(7+|X|)\varepsilon'] && \text{(by Step 3)} \\ &\geq u^\infty(\varphi^\infty(\tilde{s}_i|\tilde{\theta})|\tilde{s}_i,\tilde{\theta}) - [\alpha+(8+|X|)\varepsilon'] && \text{(by Step 1).} \end{aligned}$$

In the opposite direction, we have

$$\begin{aligned} u^\infty(\varphi^\infty(\tilde{s}_i|\tilde{\theta})|\tilde{s}_i,\tilde{\theta}) &\geq u^\infty(\mathrm{GRP}_i(\tilde{s}^n)|\tilde{s}_i,\tilde{\theta}) - (1+|X|)\varepsilon' && \text{(by Steps 3)} \\ &\geq u_i^n(\mathrm{GRP}_i(\tilde{s}^n)|\tilde{s}^n) - (2+|X|)\varepsilon' && \text{(by Steps 1).} \end{aligned}$$

These two inequalities provides the following expression:

$$\max_i |u_i^n(\mathrm{GRP}_i(\tilde{s}^n)|\tilde{s}^n) - u^\infty(\varphi^\infty(\tilde{s}_i|\tilde{\theta})|\tilde{s}_i,\tilde{\theta})| < \alpha+\varepsilon.$$

In terms of revenues, we have

$$\begin{aligned} \mathrm{Rev}^{\mathrm{GRP}}(\tilde{s}^n) &\geq \frac{1}{n}\sum_i \Big[(1-\alpha-6\varepsilon')\varphi^\infty(\tilde{s}_i|\tilde{\theta}) - \varepsilon_B - t_*(\alpha+6\varepsilon')\Big] && \text{(by Steps 3 and 5)} \\ &\geq (1-\alpha-6\varepsilon')\Big(\mathrm{Rev}^\infty(\tilde{\theta}) - \varepsilon'\Big) - \varepsilon_B - t_*(\alpha+6\varepsilon') && \text{(by Step 4)} \\ &\geq \mathrm{Rev}^\infty(\tilde{\theta}) - t_*(\alpha+6\varepsilon') - \varepsilon_B - t_*(\alpha+\varepsilon) \\ &> \mathrm{Rev}^\infty(\tilde{\theta}) - 2t_*(\alpha+\varepsilon) - \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{Rev}^{\mathrm{GRP}}(\tilde{s}^n) &\leq \frac{1}{n}\sum_i \Big[(1-\alpha-6\varepsilon')\varphi^\infty(\tilde{s}_i|\tilde{\theta}) + \varepsilon_B + (\alpha+6\varepsilon')\Big] \\ &\leq \Big(\mathrm{Rev}^\infty(\tilde{\theta}) + \varepsilon'\Big) + \varepsilon_B + (\alpha+6\varepsilon') \\ &> \mathrm{Rev}^\infty(\tilde{\theta}) + \alpha + \varepsilon. \end{aligned}$$

Therefore,

$$|\mathrm{Rev}^\infty(\tilde{\theta}) - \mathrm{Rev}^{\mathrm{GRP}}(\tilde{s}^n)| < \max\{2t_*, 1\}(\alpha+\varepsilon) - \varepsilon.$$

A.3.1 Proof of Lemma 4

Take arbitrarily small $\varepsilon > 0$. We first introduce notations. Let $A = 6 + 2|X|$. Define an upper semi-continuous mapping $g : \mathcal{B} \to (0, 1]$ by

$$g(B) = \begin{cases} 2 & \text{if } |B| = 1 \\ \min_{z,z' \in B,\, z \neq z'} d(z, z') & \text{otherwise} \end{cases} \tag{93}$$

This mapping $g(B)$ represents the minimum distance between two different elements in B. Let $V = \mathbb{R}^X$ as in Section 6.2. For each $\varepsilon' \geq 0$, $B \in \mathcal{B}_f$, and $z \in B$, let

$$V_z(\varepsilon', B) = \Big\{ v \in V : u_v(z) > u_v(z') + \varepsilon' \text{ for all } z' \in B \setminus \{z\} \Big\}, \tag{94}$$

$$V(\varepsilon', B) = \bigcup\nolimits_{z \in B} V_z(\varepsilon', B), \tag{95}$$

where $u_v(x, t) = v(x) - t$. By Lemma 1 and the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon' \searrow 0} \mu_V(V(\varepsilon', B)|\theta) = 1 \tag{96}$$

for all $\theta \in \Theta$ and $B \in \mathcal{B}_f$.

Note that the (inner) probability of the following converges to 1 as $\delta \to 0$:

(i) $\mu_V(V(2A\delta, B^\infty)|\tilde{\theta}) > 1 - \varepsilon/2$ (by (96)).

(ii) $g(B^\infty(\tilde{\theta})) \geq 2\delta$ (by $g > 0$).

Therefore, there exist $\delta^* \in (0, \varepsilon)$ such that the inner probability of $A_1 = \{$(i) and (ii) hold with $\delta = \delta^*\}$ is more than $1 - \varepsilon/3$. With this δ^*, by Lemma 2, there exists N_1 such that $n \geq N_1$ implies that the inner probability of $A_2^n = \{\omega$ is δ-regular$\}$ is more than $1 - \varepsilon/3$. Further, by the weak law of large numbers, we can find sufficiently large $N_2 > N_1$ such that $n \geq N_2$ implies the inner probability of

$$A_3^n = \left\{ \frac{1}{n} \sum\nolimits_{i=1}^n 1\Big\{ v^\infty(\cdot|\tilde{s}_i, \tilde{\theta}) \in V\big(A\delta^*, B^\infty(\tilde{\theta})\big) \Big\} \geq \mu_V\big(V(A\delta^*, B^\infty)\big|\tilde{\theta}\big) - \frac{\varepsilon}{2} \right\}. \tag{97}$$

is more than $1 - \varepsilon/3$.

At last, we show that for all $n \geq N$ the following holds in the event $A^n = A_1 \cap A_2^n \cap A_3^n$:

If $i \in N_n$ is such that $v^\infty(\cdot|\tilde{s}_i, \tilde{\theta}) \in V_z(A\delta^*, B^\infty)$, then $Y^\infty(B^\infty(\tilde{\theta})|\tilde{s}_i, \tilde{\theta})$ is a singleton $\{y_i^{(\infty)}\}$ and all $y_i^{(n)} \in Y_i^n(\delta^*, B^\infty(\tilde{\theta})|\tilde{s}^n)$ are such that $\|y_i^{(\infty)} - y_i^{(n)}\|_\infty < \delta$.

Assume $\omega \in A^n$ and consider $i \in N_n$ such that $v^\infty(\cdot|\tilde{s}_i, \tilde{\theta}) \in V_z(A\delta, B^\infty)$. On one hand, it is clear from the choice of i that $u^\infty(\cdot|\tilde{s}_i, \tilde{\theta})$ has a unique maximizer $y_i^{(\infty)}$ in $B^\infty(\tilde{\theta})$; hence

$Z^\infty(B^\infty(\tilde{\theta})|\tilde{s}_i,\tilde{\theta}) = \{y_i^{(\infty)}\}$. On the other hand, for all $z' \in B^\infty(\tilde{\theta})$, $z_\delta, z'_\delta \in \Delta(X)\times\mathbb{R}$ such that $d(z, z_\delta) \le \delta$ and $d(z', z'_\delta) \le \delta$,

$$\begin{aligned}
u_i^n(z_\delta|\tilde{s}^n) &\ge u^\infty(z_\delta|\tilde{s}_i,\tilde{\theta}) - \delta && (98)\\
&\ge u^\infty(z|\tilde{s}_i,\tilde{\theta}) - (1+|X|)\delta && (99)\\
&> u^\infty(z'|\tilde{s}_i,\tilde{\theta}) + (1+|X|)\delta && (100)\\
&\ge u^\infty(z'_\delta|\tilde{s}_i,\tilde{\theta}) + \delta && (101)\\
&\ge u_i^n(z'_\delta|\tilde{s}^n). && (102)
\end{aligned}$$

Therefore, if $z^* \in Z_i^n(\delta, B^\infty|\tilde{s}^n)$ then $d(z^*, z) \le \delta$. □

A.4 Proof of Theorem 3

For each $\bar{\theta} \in \Theta$, define $g(\cdot|\bar{\theta}) : \Theta \to \mathbb{R}^L$ by

$$\Psi_{\bar{\theta}}(\theta) = \sum_{z\in B^\infty(\theta)} h(z)\psi_{\bar{\theta},z}(\theta), \tag{103}$$

where $h(\lambda, t) = \sum_{x\in X} x\lambda(x)$ and

$$\psi_{\bar{\theta},z}(\theta) = \mu\left(\bigcap\nolimits_{z\neq z'\in B^\infty(\bar{\theta})}\left\{s : u^\infty(z|s,\theta) > u^\infty(z'|s,\theta)\right\}\middle|\theta\right). \tag{104}$$

Note that ψ is continuous in θ because

$$\begin{aligned}
\lim_{\theta\to\theta^*}\psi_{\bar{\theta},z}(\theta) &= \lim_{\theta\to\theta^*}\int\prod_{z\neq z'\in B^\infty(\bar{\theta})} 1\big\{u^\infty(z|s,\theta) > u^\infty(z'|s,\theta)\big\}f(s|\theta)\mathrm{d}s && (105)\\
&= \int\prod_{z\neq z'\in B^\infty(\bar{\theta})} \lim_{\theta\to\theta^*} 1\big\{u^\infty(z|s,\theta) > u^\infty(z'|s,\theta)\big\}\cdot\lim_{\theta\to\theta^*} f(s|\theta)\cdot\mathrm{d}s && (106)\\
&= \int\prod_{z\neq z'\in B^\infty(\bar{\theta})} 1\big\{u^\infty(z|s,\theta^*) > u^\infty(z'|s,\theta^*)\big\}f(s|\theta^*)\mathrm{d}s && (107)\\
&= \psi_{\bar{\theta},z}(\theta^*). && (108)
\end{aligned}$$

Hence, $\Psi_{\bar{\theta}}(\theta)$ is continuous in θ.

Let

$$U(\alpha,\bar{\theta}) = \Psi_{\bar{\theta}}^{-1}(\{y\in\mathbb{R}^L : y \le (1+\alpha)q\}) \cap B_\alpha(\bar{\theta}), \tag{109}$$

which is an open set containing $\bar{\theta}$. For each $m = 1, 2, \ldots$, there exists a finite set $\{\bar{\theta}_1^m, \ldots, \bar{\theta}_{K(m)}^m\} \subseteq \Theta$ such that $\Theta = \bigcup_{k=1}^{K(m)} U(1/m, \bar{\theta}_k^m)$, because $\{U(1/m,\theta)\}_{\theta\in\Theta}$ is an open covering of the compact set Θ. For each m, define $\bar{\theta}^m : \Theta \to \Theta$ by $\bar{\theta}^m(\theta) = \bar{\theta}_{k(\theta)}^m$, where $k(\theta)$ is the smallest k such that $\theta \in U(1/m, \bar{\theta}_k^m)$.

For each m, construct an incentive compatible ∞-mechanism $(\varphi_m^\infty, B_m^\infty)$ as follows: $B_m^\infty(\theta) = B^\infty(\theta^m(\theta))$ and $\varphi(s|\theta) \in Z(B_m^\infty|s,\theta)$. By construction, $\Psi_{\bar{\theta}^m(\theta)}(\theta) \leq (1+\alpha)q$, which means $(\varphi_m^\infty, B_m^\infty)$ is $(1/m)$-feasible.

For each m, let $\{\varphi_m^n\}$ be the mechanism approximating $(\varphi_m^\infty, B_m^\infty)$ as constructed in Theorem 2. Starting from $(m_1, N_1) = (1,1)$, we recursively construct $m_k > m_{k-1}$ and $N_k > N_{k-1}$ so that for all $n > N_k$, φ^{n,m_k} satisfies all four conditions (i)-(iv) with $\varepsilon = 1/k$. Finally define $\{\varphi^n\}$ by $\varphi^n = \varphi^{n,m_{k(n)}}$ where $k(n)$ is the smallest k such that $n > N_k$.

A.5 Proof of Theorems 4 and 5

The proofs for Theorems 4 and 5 are almost identical. We first present the proof of Theorem 5, and then we discuss how to modify the proof to show Theorem 4.

A.5.1 Proof of Theorem 5

Let $\{\varphi^n\}$ be the GRP used in the proof of Theorem 3, which clearly is asymptotically envy-free by construction. We establish its asymptotic efficiency. Let $\varepsilon' = \min\{\varepsilon/2, \varepsilon\cdot q_*/(10\bar{x}), 1\}$, where $q_* = \min_\ell q_\ell$.

Step 1: Approximation of distribution. Recall $\sup_\theta \|P^n(\theta) - \theta\| \to 0$ as $n \to \infty$. Also note that f is uniformly continuous because it is continuous on a compact domain. From these facts, we can find sufficiently large N_1 such that for all $n \geq N_1$,

$$|f(s|\theta) - f(s|\theta^P)| < \frac{\varepsilon'}{\bar{x}(1+\mathfrak{m}(S))} \tag{110}$$

for all $s \in S$, $\theta^P \in P^n(\Theta)$, and $\theta \in P^{n,-1}(\theta^P)$.

Step 2: "Irregular" signals are unlikely. Find sufficiently small $\varepsilon_v \in (0,1)$ such that $\mu(T^3(\theta^P)|\theta^P) < \varepsilon'$ for all $\theta^P \in P^n(\Theta)$, where

$$T^1(\theta^P) = \{s \in S : \max_x v^\infty(x|s,\theta^P) > v^\infty(\varphi^\infty(s|\theta^P)|s,\theta^P) + \varepsilon_v\}, \tag{111}$$

$$T^2(\theta^P) = \{s \in S : \max_x v^\infty(x|s,\theta^P) = v^\infty(\varphi^\infty(s|\theta^P)|s,\theta^P) > \max_{x \neq x^*(s)} v^\infty(x|s,\theta^P) + \varepsilon_v\}, \tag{112}$$

and $T^3(\theta^P) = S \setminus (T^1(\theta^P) \cup T^2(\theta^P))$. This implies

$$\mu(T^3(\theta^P)|\theta) \leq \mu(T^3(\theta^P)|\theta^P) + \int |f(s|\theta) - f(s|\theta^P)| \mathrm{d}s < 2\varepsilon' \tag{113}$$

for all $\theta \in \Theta$ such that $\theta^P = P^n(\theta)$. Let $\Theta_+^P = \{\theta^P \in P^n(\Theta) : \mu(T^1(\theta^P)|\theta^P) > 0\}$ and $\Theta_0^P = \{\theta^P \in P^n(\Theta) : \mu(T^1(\theta^P)|\theta^P) = 0\}$. Note that $\mu(T^1(\theta^P)|\theta) = 0$ is equivalent to $\mu(T^1(\theta^P)|\theta) = 0$ so long as $P^n(\theta) = \theta^P$, because $\mu(\theta)$ and $\mu(\theta^P)$ are mutually absolutely continuous.

Step 3: Weak law of large numbers. Take sufficiently large $N_3 > N_1$ such that for all integer $n \geq N_3$, the following event $A^n_{3,1}$ occurs with a probability more than $1 - \varepsilon'$:

$$\sum_{i=1}^{n} 1_{T^1(\theta^P)}(\tilde{s}_i) > 0 \qquad \text{if } \theta^P \in \Theta^P_+ \tag{114}$$

$$\sum_{i=1}^{n} 1_{T^1(\theta^P)}(\tilde{s}_i) = 0 \qquad \text{if } \theta^P \in \Theta^P_0 \tag{115}$$

$$\sum_{i=1}^{n} 1_{T^3(\theta^P))}(\tilde{s}_i) < 3\varepsilon' n \tag{116}$$

$$\sum_{i=1}^{n} \mathbb{E}\left[\varphi^\infty_\ell\big(s\big|\theta^P\big)\right] > n(1-2\varepsilon')q_\ell \quad \text{for all } \ell \in \mathcal{L}(\theta^P) \tag{117}$$

where $\theta^P = P^n(\tilde{\theta})$ and $\mathcal{L}(\theta^P)$ is the set of ℓ such that

$$\int \mathbb{E}\left[\varphi^\infty_\ell\big(s\big|\theta^P\big)\right] f(s|\theta^P)\mathrm{d}s = q_\ell \tag{118}$$

The existence of N_3 follows from Chebyshev's inequality conditional on $\tilde{\theta}$. Note that the following properties hold almost surely ((120) is due to Lemma 1):

$$\tilde{s}_i \neq \tilde{s}_j \quad \text{for all } i, j \in N_n \text{ s.t. } i \neq j \tag{119}$$

$$\arg\max_{z \in B^\infty(\theta^P)} v^\infty(z|\tilde{s}_i, \theta^P) \text{ is a singleton.} \tag{120}$$

Therefore, the event $A^n_{3,2}$ that (114)–(117) and (120)–(119) hold has the probability $\mathbb{E}(A^n_{3,2}) = \mathbb{E}(A^n_{3,1}) > 1 - \varepsilon'$.

Step 4: Application of Theorem 3 and Lemma 2. Let $\varepsilon_a = \min\{\varepsilon'\varepsilon_v/2, \varepsilon_v\}$. Take sufficiently large $N_4 > N_3$ such that the conditions of Theorem 3 and Lemma 2 are satisfied with ε_a in a measurable set E^n_4.

From now on, we fix $n \geq N_4$ and $\omega \in A^n_{3,2} \cap A^n_4$. Let $s_i = \tilde{s}_i(\omega)$, $\theta^P = P^n(\tilde{\theta}(\omega))$, and $N^k = \{i \in N_n : s_i \in T^k(\theta^P)\}$ for each $k = 1, 2, 3$. Since $\mathbb{P}(A^n_{3,2} \cap A^n_4) > 1 - 2\varepsilon' > 1 - \varepsilon$, to complete the proof, it suffices to show the ε-efficiency of $\varphi^n(s^n)$.

Step 5: Case with $\theta^P \in \Theta^P_0$. Let $x^*_i = \arg\max_{x \in X} v^n_i(x|s^n)$. Since N^1 is empty,

$$\begin{aligned}
v^n_i(\varphi^n_i(s^n)|s^n) &> v^\infty(\varphi^\infty(s_i|\theta^P)|s_i, \theta^P) - \varepsilon_a && \text{(by Step 4)} && (121)\\
&\geq v^\infty(x^*_i|s_i, \theta^P) - \varepsilon_v - \varepsilon_a && \text{(by } i \notin N^1) && (122)\\
&> v^n_i(x^*_i|s^n) - \varepsilon_v - 2\varepsilon_a && \text{(by Step 4)} && (123)\\
&\geq v^n_i(z_i|s^n) - \varepsilon && && (124)
\end{aligned}$$

for all $z_i \in \Delta(X)$. Therefore, $\varphi^n(s^n)$ is ε-efficient at s^n.

Step 6: Case with $\theta^P \in \Theta^P_+$. We then assume $\theta^P \in \Theta^P_+$. Let $\hat{\lambda} \in \Delta(X^n)$ be such that

(i) $\hat{\lambda}$ is ε-feasible, and

(ii) $v_i(\hat{z}_i|\tilde{s}^n) \geq v_i(\varphi^n_i(\tilde{s}^n)|\tilde{s}^n)$ for all $i \in N_n$,

where $\hat{z}_i$ is the i-th marginal distribution of $\hat{\lambda}$. Let $(z^*_1, \ldots, z^*_n)$ be such that

$$z^*_i = \begin{cases} (1-\varepsilon')\hat{z}_i + \varepsilon' x^*(s_i) & \text{if } i \in N_1 \\ x^*(s_i) & \text{if } i \in N_2 \cup N_3, \end{cases} \tag{125}$$

where $x^*(s_i) = \arg\max_{x \in X} v^\infty(x|s_i, \theta^P)$.

First note that the sum of the expected values of $(z^*_1, \ldots, z^*_n)$ is less than that of $(\varphi^\infty(s_1|\theta^P), \ldots, \varphi^\infty(s_n|\theta^P))$ in dimensions $\ell \in \mathcal{L}(\theta^P)$: Since $|N^3| < 3\varepsilon' n$, we have

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}[z^*_{i,\ell}] &\leq \sum_{i \in N^1} \big(\mathbb{E}[\hat{z}_{i,\ell}] + \varepsilon' \cdot \bar{x}\big) + \sum_{i \in N^2} \mathbb{E}[\hat{z}_{i,\ell}] + \sum_{i \in N^3} \big(\mathbb{E}[\hat{z}_{i,\ell}] + \bar{x}\big) && (126) \\ &\leq (1-\varepsilon) n q_\ell + 4\varepsilon' n \cdot \bar{x} && (127) \\ &< (1-\varepsilon/2) n q && (128) \\ &\leq \sum_{i=1}^n \mathbb{E}\big[\varphi^\infty_\ell(s_i|\theta^P)\big]. && (129) \end{aligned}$$

Second note that agents i in N_1 prefer z^*_i to $\varphi^\infty(s_i|\theta^P)$ at θ^P. To show it, let $\bar{v}^\infty(s_i) = \max v^\infty(x|s_i, \theta)$. Then

$$\begin{aligned} v^\infty(z^*_i|s_i, \theta^P) &> v^n_i(z^*_i|s^n) - \varepsilon_a && \text{(by Step 4)} && (130) \\ &\geq (1-\varepsilon') v^n_i(z_i|s^n) + \varepsilon' \bar{v}^\infty(s_i) - \varepsilon_a && \text{(by (125))} && (131) \\ &\geq (1-\varepsilon') v^n_i(\varphi^\infty(s_i|\theta^P)|s^n) + \varepsilon' \bar{v}^\infty(s_i) - \varepsilon_a && \text{(by (ii))} && (132) \\ &> (1-\varepsilon') v^\infty(\varphi^\infty(s_i|\theta^P)|s_i, \theta^P) + \varepsilon' \bar{v}^\infty(s_i) - 2\varepsilon_a && \text{(by Step 4)} && (133) \\ &> v^\infty(\varphi^\infty(s_i|\theta^P)|s_i, \theta^P) + \varepsilon' \varepsilon_v - 2\varepsilon_a && \text{(by } i \in N_1) && (134) \\ &\geq v^\infty(\varphi^\infty(s_i|\theta^P)|s_i, \theta^P) && \text{(by } 2\varepsilon_a \leq \varepsilon' \varepsilon_v). && (135) \end{aligned}$$

Finally, agents $i \in N_2 \cup N_3$ at least weakly prefer z^*_i to $\varphi^\infty(s_i|\theta^P)$ simply because $z^*_i = x^*(s_i, \theta^P)$ is optimal for them. That is,

$$v^\infty(z^*_i|s_i, \theta^P) \geq v^\infty(\varphi^\infty(s_i|\theta^P)|s_i, \theta^P). \tag{136}$$

To complete the proof, we derive a contradiction by "embedding" z^* to the continuous economy with θ^P. Let $U(r, s) = \{s' \in S : \|s - s'\| < r\}$ be the r-open ball of s. Take sufficiently small $r^* > 0$ such that $\{U(r^*, s_i)\}_{i=1}^n$ are disjoint and

$$\varphi^\infty(s_i|\theta^P) = \operatorname*{arg\,max}_{z \in B^\infty(\theta^P)} v^\infty(z|s'_i, \theta^P) \tag{137}$$

$$v^\infty(z^*_i|s'_i, \theta^P) > \max_{z \in B^\infty(\theta^P)} v^\infty(z|s'_i, \theta^P) \quad \text{if } i \in N_1 \tag{138}$$

for all $i \in N_n$ and $s_i' \in U_i \equiv U(r^*, s_i)$. Such r^* exists by (119), (120), (135), and the continuity of v^∞. Let $\bar{a} = \min_i \mathfrak{m}(U_i)$. For each $a \in (0, \bar{a})$, define a measurable mapping $z^a : S \to \Delta(X)$ by

$$z^a(s) = \begin{cases} \dfrac{a}{\mathfrak{m}(U_i)} \cdot z_i^* + \left(1 - \dfrac{a}{\mathfrak{m}(U_i)}\right) \cdot \varphi^\infty(s|\theta^P) & \text{if } s \in B_i \text{ with some } i \in N_n \\ \varphi^\infty(s|\theta^P) & \text{otherwise.} \end{cases} \tag{139}$$

By (129), $z^{**} = z^a$ is a feasible plan for some sufficiently small $a \in (0, \bar{a})$. Also, by (137) and (138),

$$v^\infty(z^{**}(s)|s, \theta) \geq v^\infty(\varphi^\infty(s|\theta)|s, \theta) \tag{140}$$

for all $s \in S$ and the inequality is strict for all $s \in B_i$ such that $i \in N^1$. However, this contradicts the efficiency of φ^∞. □

A.5.2 Proof of Theorem 4

In the proof of Theorem 4, using Steps 1–5 of the above proof, we can can simplify the argument in Step 6. Let $\lambda' \in \Delta(X^n)$ be a lottery on X^n such that

$$\frac{1}{n}\sum_{i=1}^n v_i^n(z_i'|s^n) > \frac{1}{n}\sum_{i=1}^n v_i^n(\varphi_i^n(s^n)|s^n) + \varepsilon, \tag{141}$$

where $(z_1', \ldots, z_n')$ be its marginal distributions. Let $z_i^* = (1 - \varepsilon')z_i' + \varepsilon' 0$ for each i. Then, the condition for the feasibility holds:

$$\sum_{i=1}^n \mathbb{E}[z_i^*] \leq (1 - \varepsilon)nq < \sum_{i=1}^n \mathbb{E}\big[\varphi^\infty(s_i|\theta^P)\big] \tag{142}$$

Also,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n v_i^n(z_i^*|s^n) &\geq \frac{1}{n}\sum_{i=1}^n v_i^n(z_i'|s^n) - \varepsilon' && (143) \\ &> \frac{1}{n}\sum_{i=1}^n v_i^n(\varphi_i^n(s^n)|s^n) + \varepsilon - \varepsilon' && (144) \\ &> \frac{1}{n}\sum_{i=1}^n v^\infty(\varphi_i^\infty(s_i|\theta^P)|s_i, \theta^P) + \varepsilon - \varepsilon' - \varepsilon_a && (145) \\ &> \frac{1}{n}\sum_{i=1}^n v^\infty(\varphi_i^\infty(s_i|\theta^P)|s_i, \theta^P). && (146) \end{aligned}$$

Finally, we use the argument of the last paragraph of Step 6.

A.6 Proof of Proposition 1

Fix $\lambda_V \in \Delta^{\mathrm{NI}}(V)$. By the definition of $\Delta^{\mathrm{NI}}(V)$, the demand correspondence $D(p;v)$ is a singleton almost everywhere in λ_V for all $p \in [0,\infty)^L$. Thus the aggregate demand correspondence

$$A(p) = \int D(p;v)\mathrm{d}\lambda_V(v) \tag{147}$$

is indeed a function of $p \in [0,\infty)^L$. Furthermore, $A(p)$ is continuous because

$$\lim_{p\to p'} A(p) = \int \lim_{p\to p'} D(p;v)\mathrm{d}\lambda(v) = \int D(p';v)\mathrm{d}\lambda(v) = A(p') \tag{148}$$

by Lebesgue's dominated convergence theorem.

Find $\bar{p} > 0$ such that $A_\ell(p) < q_\ell$ whenever $p_\ell > \bar{p}$. Such $\bar{p}$ exists because $\lambda_V\{v \in V : \|v\|_\infty \geq a\}$ converges to 0 as $a \to \infty$.[18] For each $p \in [0, \bar{p}+\bar{x}]^L$, define

$$\phi(p) = p + A(p) - q. \tag{149}$$

The image of ϕ is a subset of $D = [-\bar{q}, \bar{p}+\bar{x}]^L$: If $p_\ell \leq \bar{p}$ then $\phi_\ell(p) \leq p_\ell + A_\ell(p) \leq \bar{p}+\bar{x}$, and $p_\ell > \bar{p}$ implies $\phi_\ell(p) \leq p_\ell \leq \bar{p}+\bar{x}$ since $A_\ell(p) < q_\ell$. From the function ϕ, we define a mapping $\psi : D \to D$ by

$$\psi(p) = \phi(p \vee 0).^{19} \tag{150}$$

This mapping is continuous because so is $A(p)$. Therefore, the mapping ψ has a fixed point p^F by Brouwer's fixed point theorem.[20]

We prove that $p^W = p^F \vee 0$ constitutes a Walrasian equilibrium together with a consumption function $x^W(v)$ such that $x^W(v) \in D(p^W;v)$. Since $p^W \geq p^F$, the market clearing condition is satisfied:

$$\int x^W \mathrm{d}\lambda = A(p^W) = q - p^W + \phi(p^W) = q - (p^W - p^F) \leq q. \tag{151}$$

The inequality is strict in the ℓ-th dimension only when $p^W_\ell > p^F_\ell$, which occurs only when $p^W_\ell = 0$. Therefore, the pair (x^W, p^W) comprises a Walrasian equilibrium for economy λ_V. $\square$

[18] This is shown, for example, by Lebesgue's dominated convergence theorem: $\lambda_V\{v \in V : \|v\|_\infty \geq a\} = \int 1\{\|v\|_\infty \geq a\} \to \int \lim_{a\to\infty} 1\{\|v\|_\infty \geq a\} = \int 0 = 0$ as $a \to \infty$.

[19] Here $p' = p \vee 0$ is the dimension-wise maximum of p and the L-dimensional zero vector; that is, $p'_\ell = \max\{p_\ell, 0\}$.

[20] This fixed point method also appears in Azevedo et al. (2012, Theorem 1) and Budish et al. (2012, Theorem 6).

A.7 Proof of Proposition 2

Define the excess demand correspondence

$$Z(p) = \int \mathbb{E}[Z(p; v)] \mathrm{d}\lambda(v) - q, \tag{152}$$

where the integral is the Aumann integral and $\mathbb{E}[Z(p; v)] = \{\mathbb{E}[z] : z \in Z(p; v)\}$. Then, $Z(p)$ is a non-empty, compact, and convex-valued upper hemicontinuous correspondence (Debreu, 1967, Section 6), The rest of the proof is identical to the proof of Theorem 6 in Budish et al. (2012). □

A.8 Proof of Proposition 3

The no-envy property is a direct consequence of utility maximization. To prove efficiency, assume that z' is a feasible plan such that $\lambda(V_+) = 1$ and $\lambda(V_{++}) > 0$, where $V_+ = \{v : v(z'(v)) \geq v(z^{\mathrm{HZ}}(v))\}$ and $V_{++} = \{v : v(z'(v)) > v(z^{\mathrm{HZ}}(v))\}$. As a consequence of utility maximization, $p^{\mathrm{HZ}} \cdot \mathbb{E}[z'(v)] \geq p^{\mathrm{HZ}} \cdot \mathbb{E}[z^{\mathrm{HZ}}(v)]$ for all $v_+ \in V_+$, and when $v \in V_{++}$, this inequality is strict. By integrate the above inequality with respect to v,

$$p^{\mathrm{HZ}} \cdot q \geq p^{\mathrm{HZ}} \cdot \int \mathbb{E}[z'(v)] \mathrm{d}\lambda(v) > p^{\mathrm{HZ}} \cdot \int \mathbb{E}[z^{\mathrm{HZ}}(v)] \mathrm{d}\lambda(v) = p^{\mathrm{HZ}} \cdot q. \tag{153}$$

This is a contradiction. □

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