Altruistic Kidney Exchange*

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1 Introduction

Transplantation is preferred to dialysis due to both medical and economic reasons by end-stage kidney disease patients. It is illegal to buy or sell a human body organ, including a kidney, in most of the countries in the world.[1] Moreover, a transplanted kidney from a live donor survives longer than that from a deceased donor (Mandal et al., 2003). Hence, live donation is the preferred solution for a patient. There are two kidneys in human body, one is often enough for survival. With the advancement of medical techniques, live donation has been the increasing source of donations. Usually, live donors are friends and family members of a patient. However, a patient has about 47% chance to receive a willing live-donor's kidney (see Roth, Sönmez, and Ünver (2005a) simulations), because of blood-type incompatibility or antibodies to one of the donor's proteins known as tissue-type incompatibility.

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[1]The National Organ Transplant Act of 1984 makes it illegal to buy or sell a kidney in the US and makes donation the only viable option for kidney transplantation. It is also illegal to buy or sell a body organ in rest of the western world World Health Organization condemns the practice of organ sales.

Humans have four blood-types "O", "A", "B", and "AB", named according to the existence or non-existence of "A" or "B" type blood proteins in blood cells. While "O" donors are blood-type compatible with all recipients, "A" donors are only blood-type compatible with "A" or "AB" recipients, "B" donors are only blood-type compatible with "B" or "AB" recipients, and "AB" donors only blood-type compatible with "AB" recipients.

Even if a donor is blood-type compatible with a recipient, she may not donate a kidney to him due to tissue-type incompatibility. Zenios, Woodle, and Ross (2001) report that there is about 11% chance for this to happen for a random patient and blood-type compatible donor.

Rapaport (1986) proposed "paired kidney donations" to be performed between two such incompatible patient-donor pairs (blood-type incompatible or tissue-type incompatible): the donor in each pair gives a kidney to the other pair's compatible patient. In the 2000's, New England, Ohio, and Johns Hopkins transplant programs in the US started conducting ad-hoc live donor kidney exchange operations. The initial hurdle in organizing kidney exchanges was the lack of mechanisms to clear the market in an efficient and incentive compatible manner. Roth, Sönmez, and Ünver (2004) proposed the first such clearance mechanism.

One important aspect of kidney exchanges is that regardless of the number of pairs participating in one exchange, all transplants in an exchange should be conducted simultaneously. Otherwise, one ore more of the live donors whose patients receive a kidney in the previously conducted part of an exchange may back out from the future donations of the same exchange. Since kidney donations are gifts, the donor can change her mind at any moment in time prior to the actual transplant, and it is not legal to write a contract that forces a donor for future donations.

This practice naturally places an upper limit on the number of kidney transplants that can be conducted simultaneously. Based on this important restriction, Roth, Sönmez, and Ünver (2005b) focused on only exchanges consisting of 2 pairs. As the two of the coauthors of the above study, we analyzed the structure of Pareto-efficient matchings when only two-way exchanges are possible and patients are indifferent among compatible donors. Then we proposed incentive-compatible and Pareto-efficient mechanisms based on two polar fairness approaches: a priority-based mechanism that matches pairs based on an exogenous priority structure and a probabilistic egalitarian mechanism that matches all pairs as equal probability as possible without violating medical compatibility constraints and Pareto-efficiency.[2]

[2]New England Program for Kidney Exchange (NEPKE) is the first US kidney exchange program that started to

Since its conception, one important aspect of the practical kidney exchange paradigm has been almost exclusively having incompatible pairs as participants in kidney exchange, while the donors of compatible pairs directly donate to their paired-recipients.

For a two-way exchange to go through, the donor of each pair at least to be blood-type compatible with the recipient of the other pair. Thus, two blood-type incompatible pairs can never participate in a two-way kidney exchange (Roth, Sönmez, and Ünver, 2007; Ünver, 2010). When compatible pairs do not participate in exchange, a blood-type compatible pair is available only if the pair is tissue-type incompatible. Thus, in theory, we can boost the number of exchanges by introducing compatible pairs into exchange.

Moreover, in the US, doctors generally assume that two live-donor kidneys survive almost the same duration regardless of the genetic distance between the donor and the recipient's certain DNA proteins (Gjertson and Cecka, 2000; Delmonico, 2004). Although the positive correlation between genetic closeness of the donor to the recipient and transplant organ survival is unanimously accepted for deceased-donor transplants, there is no conclusive empirical evidence for such a positive correlation for live-donor kidney transplants. Controlling for other properties such as medical background, age etc., many compatible pairs could be willing to participate in exchange. This idea is not new, and was first proposed by Ross and Woodle (2000). Later this idea received little attention until Roth, Sönmez, and Ünver (2004, 2005a), which proposed the same concept in mechanism design.

In this paper, we consider a kidney exchange model in which both compatible and incompatible pairs available for kidney exchange. In a kidney exchange only 2 pairs can participate and patients are indifferent across all compatible donors. We aim to understand the structure of Pareto-efficient matchings in this framework. In Roth, Sönmez, and Ünver (2004, 2005a), we developed a theory for

implement mechanisms for kidney exchange, and was established in 2004 through the our collaboration with Alvin E. Roth, and medical professionals Francis Delmonico and Susan Saidman. NEPKE started to implement a version of the priority mechanism in 2004 proposed in the above paper (Roth, Sönmez, and Ünver, 2005a) It was followed by the Johns Hopkins Kidney Exchange Program (Segev et al., 2005), which adopted a similar algorithm due to Edmonds (1965) as earlier proposed by Roth, Sönmez, and Ünver (2005b). Later more complicated exchange scenarios were embedded in these programs and a larger inter-regional program, Alliance for Paired Donation (APD) was established in 2005 through our collaboration with Alvin E. Roth, and medical professionals Dr. Michael Rees and Jon Kopke. In late 2009, a pilot national program will be started under the provision of United Network for Organ Sharing (UNOS), the federal entity that also foresees deceased donor donations. This program will use a similar matching mechanism to the APD mechanism.

two-way kidney exchanges in a similar environment where the only difference is only incompatible pairs can participate in exchange. We proposed Pareto-efficient and incentive compatible mechanisms for this environment. In this previous framework, most underlying structure has been readily available in the combinatorial optimization literature through the works of Gallai (1963, 1964); Edmonds (1965); and many others (see Lovasz and Plummer (1986) and Korte and Vygen (2002) for a comprehensive survey of such work). The key result in this literature was the Gallai (1963, 1964) - Edmonds (1965) Decomposition Lemma, which gives a clear structure of Pareto-efficient matchings. However, in the present model that we introduce, a result that we can borrow from previous literature does not exist. It also turns out that our model is not just a trivial extension of the previous model so that we can invoke some of the earlier results proven for the simpler model in our more advanced framework.

Thus, our aim in this paper is to develop the necessary machinery and both underlying mathematical and economic structure that governs this problem. First, we prove that under any two Pareto-efficient matchings the same number of patients receives transplant (Proposition 1). Moreover, whenever A is the set of compatible pairs and J is the set of incompatible pairs that are participating in two-way exchanges in a Pareto-efficient matching, there is a Pareto-efficient matching that matches exactly pairs in $A \cup J$ in two-way exchanges (Proposition 2). These two results imply that we can find a Pareto-efficient matching that matches the same incompatible pairs in two-way exchanges as another given Pareto-efficient matching, and uses the least number of compatible pairs possible to construct any other Pareto-efficient matching (Corollary 1). This result is particularly important, since the policy maker may try to minimize the number of compatible pairs participating in exchanges, and this policy will have no particular effect in determining which incompatible pairs to match.

We continue by proposing a priority mechanism that is compatible with the current practice in New England Kidney Exchange Program, and incorporates compatible pairs to the system.

Later, we analyze the structure of Pareto-efficient matchings. We derive a decomposition result that extends the GED Lemma to our framework. We decompose the set of pairs into 3 groups as the set of underdemanded pairs, the set of overdemanded pairs, and the set of perfectly matched pairs. An underdemanded pair is an incompatible pair that remains unmatched under at least one Pareto-efficient matching. An overdemanded pair is a pair that is not underdemanded but mutually compatible with an underdemanded pair. A perfectly matched pair is a pair that is neither underdemanded nor overdemanded. Our decomposition result (Proposition 4) shows that overdemanded pairs are always matched at a Pareto-efficient matching with underdemanded pairs. If we remove the overdemanded pairs from the problem, the remaining pairs are grouped in components. A component of the subproblem is a minimal set of pairs that are not mutually compatible with any pair (of the subproblem) outside of this set. We show that, a pair is underdemanded if and only if it is a member of such a component that consists of odd number of incompatible pairs (Proposition 4). We further prove that under any Pareto-efficient matching all but one of the members of such an odd-numbered component of incompatible pairs are matched with each other while the remaining member is unmatched or gets matched with an overdemanded pair.

2 The Model

A **pair** consists of a patient and a donor. A pair is **compatible** if the donor of the pair can medically donate her kidney to the patient of the pair, and **incompatible** otherwise. Let N_I be the set of incompatible pairs and N_C be the set of compatible pairs. Let $N = N_I \cup N_C$ be the set of all pairs. The donor of pair x is compatible with the patient of pair y if the donor of pair x can medically donate a kidney to the patient of pair y. Two distinct pairs $x, y \in N$ are **mutually compatible** if the donor of pair x is compatible with the patient of pair y and the donor of pair y is compatible with the patient of pair x.

For any pair $x \in N$, let $\succsim_x$ denote its preferences over N. Let $\succ_x$ denote the strict preference relation and $\sim_x$ denote the indifference relation associated with $\succsim_x$. Preferences of a pair is dictated by the patient of the pair who is indifferent between all compatible kidneys and who strictly prefers any compatible kidney to any incompatible kidney. In addition, the patient of an incompatible pair strictly prefers remaining unmatched (i.e. keeping her donor's incompatible kidney) to any other incompatible kidney. Therefore for any incompatible pair $i \in N_I$,

- $x \sim_i y$ for distinct $x, y \in N$ with a compatible donor for the patient of pair i,
- $x \succ_i i$ for any $x \in N$ with a compatible donor for the patient of pair i,
- $i \succ_i x$ for any $x \in N$ without a compatible donor for the patient of pair i,

and for any compatible pair $c \in N_C$,

- $x \sim_c y$ for distinct $x, y \in N$ with a compatible donor for the patient of pair c,

- $c \succ_c x$ for any $x \in N$ without a compatible donor for the patient of pair c.

Throughout the paper we assume that only two-way exchanges are feasible where at least one of the pairs is incompatible.[3] A two-way exchange is **ordinary** if it is an exchange between two incompatible pairs that are mutually compatible. A two-way exchange is **altruistically unbalanced** if it is an exchange between an incompatible and a compatible pair that are mutually compatible.

The **feasible exchange matrix** $R = [r_{x,y}]_{x,y \in N}$ identifies all feasible exchanges where

$$r_{x,y} = \begin{cases} 1 & \text{if } y \in N \setminus \{x\},\ x, y \text{ are mutually compatible, and } x \text{ or } y \in N_I \\ 0 & \text{otherwise.} \end{cases}$$

For any $x, y \in N$ with (1) $r_{x,y} = 1$ and (2) either $x \in N_I$ or $y \in N_I$, we refer the pair (x, y) as a **feasible exchange**.

An **altruistic kidney exchange problem** (or simply a **problem**) (N, R) consists of a set of pairs and its feasible exchange matrix.

A **matching** is a set of mutually exclusive feasible exchanges. Formally, given a set N of pairs a matching is a set $\mu \subseteq 2^{N^2}$ such that

1. $(x, y) \in \mu$ and $(x, y') \in \mu$ implies $y = y'$,

2. $(x, y) \in \mu$ and $(x', y) \in \mu$ implies $x = x'$, and

3. $(x, y) \in \mu$ implies $r_{x,y} = 1$ and either $x \in N_I$ or $y \in N_I$.

Here $(x, y) \in \mu$ means that the patient of each pair receives a kidney from the donor of the other pair. Let $\mathcal{M}(N, R)$ denote the set of all matchings for a given problem (N, R).[4]

For any $\mu \in \mathcal{M}(N, R)$ and $(x, y) \in \mu$, define $\mu(x) \equiv y$ and $\mu(y) \equiv x$. Here x and y are **matched** with each other in μ. For any $\mu \in \mathcal{M}(N, R)$ and $x \in N$ with no $y \in N \setminus \{x\}$ such that $(x, y) \in \mu$, define $\mu(x) \equiv x$. Here x is **unmatched** in μ. For any matching μ, let M^μ denote the set of pairs that are matched in μ. Formally,

$$M^\mu = \{x \in N : \mu(x) \neq x\}.$$

[3]Clearly there is no benefit from an exchange between two compatible pairs in our model.

[4]The ordering of pairs in a feasible exchange is not important, thus $(x, y) = (y, x)$ in our notation.

Observe that an incompatible pair receives a transplant in a matching μ only if it is matched in μ whereas a compatible pair receives a transplant whether it is matched or not. For any matching μ, let T^{μ} denote the set of all pairs who receive a transplant in μ. Formally,

$$T^{\mu} = \{x \in N_I : \mu(x) \neq x\} \cup N_C.$$

3 Pareto-Efficient Matchings

Throughout this section, fix a problem (N, R). For any $\mu, \nu \in \mathcal{M}$, μ **Pareto-dominates** ν if $\mu(x) \succsim_x \nu(x)$ for all $x \in N$ and $\mu(x) \succ_x \nu(x)$ for some $x \in N$. A matching $\mu \in \mathcal{M}$ is **Pareto-efficient** if there exists no matching that Pareto-dominates μ. Let $\mathcal{E} \subseteq \mathcal{M}$ be the set of Pareto-efficient matchings.

When there are no compatible pairs, it is well-known that the same number of incompatible pairs are matched at each Pareto efficient matching. In our model what is critical is who receives a transplant (rather than who is matched). In our first result we show that the number of the pairs who receive a transplant is the same in any Pareto-efficient matching.

Proposition 1 *For any two Pareto-efficient matchings $\mu, \nu \in \mathcal{E}$, we have $|T^{\mu}| = |T^{\nu}|$.*

Proof of Proposition 1. We will prove the proposition by showing that if two matchings $\mu, \nu \in \mathcal{M}$ are such that $|T^{\mu}| > |T^{\nu}|$, then there exists a matching that Pareto-dominates ν. Let $\mu, \nu \in \mathcal{M}$ be such that $|T^{\mu}| > |T^{\nu}|$. Let $a_0 \in T^{\mu} \setminus T^{\nu}$. Since patients of compatible pairs always receive a transplant, $a_0 \in N_I$ and therefore $a_0 \in M^{\mu}$. Construct the sequence $\{a_0, a_1, \ldots, a_k\} \subseteq M^{\mu} \cup M^{\nu}$ as follows:

$$\begin{aligned} a_1 &= \mu(a_0), \\ a_2 &= \nu(a_1), \\ &\vdots \\ a_k &= \begin{cases} \mu(a_{k-1}) & \text{if } k \text{ is odd} \\ \nu(a_{k-1}) & \text{if } k \text{ is even} \end{cases} \end{aligned}$$

and where the last element of the sequence, a_k, is unmatched either in μ or in ν (i.e. $a_k \in (M^{\mu} \setminus M^{\nu}) \cup (M^{\nu} \setminus M^{\mu})$). Observe that by construction a_0 is matched in μ but not in ν, whereas $a_1, \ldots, a_{k-1}$

are all matched in both μ and ν. Also observe that $(a_\ell, a_{\ell+1})$ is a feasible exchange for any $\ell \in \{0, 1, \ldots, k-1\}$.

There are three cases to consider:

Case 1. $a_k \in T^\nu \setminus T^\mu$:

This case, indeed, does not help us to construct a matching that Pareto-dominates ν. However since

(i) $|T^\mu| > |T^\nu|$, and

(ii) any pair that is not at the two ends of the sequence receives a transplant in both μ and ν,

there exists $a_0 \in T^\mu \setminus T^\nu$ such that the last element of the above constructed sequence a_k is such that $a_k \notin T^\nu \setminus T^\mu$. Hence Case 1 cannot cover all situations.

Case 2. $a_k \in M^\mu \setminus M^\nu$:

Since a_k is matched in μ but not in ν, k is odd. Consider the following matching $\eta \in \mathcal{M}$:

$$\eta = (\nu \setminus \{(a_1, a_2), (a_3, a_4), \ldots, (a_{k-2}, a_{k-1})\}) \cup \{(a_0, a_1), (a_2, a_3), \ldots, (a_{k-1}, a_k)\}.$$

We have $T^\eta = T^\nu \cup \{a_0, a_k\}$. Since $a_0 \notin T^\nu$, matching η Pareto-dominates matching ν.

Case 3. $a_k \in N_C$ and $a_k \in M^\nu \setminus M^\mu$:

Since a_k is matched in ν but not in μ, k is even. Consider the following matching $\eta \in \mathcal{M}$:

$$\eta = (\nu \setminus \{(a_1, a_2), (a_3, a_4), \ldots, (a_{k-1}, a_k)\}) \cup \{(a_0, a_1), (a_2, a_3), \ldots, (a_{k-2}, a_{k-1})\}.$$

Observe that a_k is matched in ν but not in η whereas a_0 is matched in η but not in ν. But since $a_k \notin N_I$, $T^\eta = T^\mu \cup \{a_0\}$ and therefore matching η Pareto-dominates matching ν.

Since there exists $a_0 \in T^\mu \setminus T^\nu$ where either Case 2 or Case 3 applies, matching ν is Pareto-inefficient. $\Diamond$

Our next result shows that the choice of compatible pairs to be matched at a Pareto efficient matching can be separated from the choice of incompatible pairs.

Proposition 2 *Let $\mu, \nu \in \mathcal{E}$ be two Pareto-efficient matchings such that $M^\mu \cap N_C = A$ and $M^\nu \cap N_I = J$. Then there exists a Pareto-efficient matching $\eta \in \mathcal{E}$ such that $M^\eta = A \cup J$.*

Proof of Proposition 2. Let μ, ν be as in the statement of the proposition. By Proposition 1, $|T^\mu \setminus T^\nu| = |T^\nu \setminus T^\mu|$. If $T^\mu = T^\nu$ then $\eta = \mu$ and we are done. Otherwise let $a_0 \in T^\mu \setminus T^\nu$. Note that $a_0 \in N_I$ (since only incompatible pairs can receive a transplant in one matching but not in another). We will construct a matching that matches a_k together with all elements of M^μ except $a_0 \in N_I$. Repeated application of this construction yields the desired matching η.

Construct the sequence $\{a_0, a_1, \ldots, a_k\} \subseteq M^\mu \cup M^\nu$ as follows:

$$
\begin{aligned}
a_1 &= \mu(a_0), \\
a_2 &= \nu(a_1), \\
&\vdots \\
a_k &= \begin{cases} \mu(a_{k-1}) & \text{if } k \text{ is odd} \\ \nu(a_{k-1}) & \text{if } k \text{ is even} \end{cases}
\end{aligned}
$$

and where the last element of the sequence, a_k, is unmatched either in μ or in ν (i.e. $a_k \in (M^\mu \setminus M^\nu) \cup (M^\nu \setminus M^\mu)$). Observe that $(a_\ell, a_{\ell+1})$ is a feasible exchange for any $\ell \in \{0, 1, \ldots, k-1\}$.

There are three cases to consider:

Case 1. k is odd:

In this case both a_0 and a_k are matched in μ, but not in ν. Consider the matching

$$\nu' = (\nu \setminus \{(a_1, a_2), (a_3, a_4), \ldots, (a_{k-2}, a_{k-1})\}) \cup \{(a_0, a_1), (a_2, a_3), \ldots, (a_{k-1}, a_k)\}.$$

By construction $M^{\nu'} = M^\nu \cup \{a_0, a_k\}$. Moreover while a_k may not be an incompatible pair, a_0 is and hence $T^\nu \subset T^{\nu'}$. Therefore ν' Pareto-dominates ν contradicting Pareto efficiency of ν.

Case 2. k is even with $a_k \in N_C$:

In this case a_k, a compatible pair, is matched in ν but not in μ. In contrast a_0, an incompatible pair, is matched in μ but not in ν. Consider the matching

$$\nu' = (\nu \setminus \{(a_1, a_2), (a_3, a_4), \ldots, (a_{k-1}, a_k)\}) \cup \{(a_0, a_1), (a_2, a_3), \ldots, (a_{k-2}, a_{k-1})\}.$$

By construction $M^{\nu'} \setminus M^{\nu} = \{a_0\}$ whereas $M^{\nu} \setminus M^{\nu'} = \{a_k\}$. Since a_0 is an incompatible pair while a_k is not, $T^{\nu} \subset T^{\nu'}$. Therefore ν' Pareto-dominates ν contradicting Pareto efficiency of ν.

Since Cases 1 and 2 each yield a contradiction, for each $a_0 \in T^{\mu} \setminus T^{\nu}$, the last element a_k of the above constructed sequence $\{a_0, a_1, \ldots, a_k\}$ should be an incompatible pair and k should be even. We next consider this final case.

Case 3. k is even with $a_k \in N_I$:

In this case a_k is matched in ν and therefore by construction $a_k \in T^{\nu} \setminus T^{\mu}$. Consider the matching

$$\mu' = (\mu \setminus \{(a_0, a_1), (a_2, a_3), \ldots, (a_{k-2}, a_{k-1})\}) \cup \{(a_1, a_2), (a_3, a_4), \ldots, (a_{k-1}, a_k)\}.$$

By construction $M^{\mu'} = (M^{\mu} \setminus \{a_0\}) \cup \{a_k\}$. So in comparison with matching μ, matching μ' matches incompatible pair a_k instead of incompatible pair a_0. Observe that $|T^{\mu'} \cap T^{\nu}| = |T^{\mu} \cap T^{\nu}| + 1$ while $M^{\mu} \cap N_C = M^{\mu'} \cap N_C$. If $|T^{\nu} \setminus T^{\mu}| = 1$, then $\eta = \mu'$ is the desired matching and we are done. Otherwise, since Case 3 is the only viable case we can repeat the same construction for any $a_0 \in T^{\mu} \setminus T^{\nu}$ to obtain the desired matching η.

$\Diamond$

In the present context, the involvement of compatible pairs in exchange is purely altruistic and it may be therefore plausible to minimize the number of compatible pairs matched at Pareto efficient matchings. An immediate corollary of Proposition 2 is that, the number of compatible pairs who exchange kidneys can be minimized without affecting the choice of incompatible pairs.

Corollary 1 *Let $\mu \in \mathcal{E}$ be such that $M^{\mu} \cap N_I = J$. Then there exists $\eta \in \mathcal{E}$ be such that $M^{\eta} \cap N_I = J$ and $|M^{\eta} \cap N_C| \leq |M^{\nu} \cap N_C|$ for any $\nu \in \mathcal{E}$.*

3.1 The Priority Mechanisms

The experience of transplant centers is mostly with the priority allocation systems used to allocate cadaver organs. New England Program for Kidney Exchange has recently adopted a variant of a priority allocation system for ordinary kidney exchanges. Priority mechanisms can be easily adopted to the present context.

Let $|N_I| = n$. A priority ordering is a one-to-one and onto function $\pi : \{1, \ldots, n\} \to N_I$. Here incompatible pair $\pi(k)$ is the k^{th} highest priority pair for any $k \in \{1, \ldots, n\}$.

For any problem, the **priority mechanism** induced by π picks any matching from a set of matchings $\mathcal{E}_\pi^n$ which is obtained by refining the set of matchings in n steps as follows:

- Let $\mathcal{E}_\pi^0 = \mathcal{M}$ (i.e. the set of all matchings).
- In general for $k \leq n$, let $\mathcal{E}_\pi^k \subseteq \mathcal{E}_\pi^{k-1}$ be such that

$$\mathcal{E}_\pi^k = \begin{cases} \left\{\mu \in \mathcal{E}_\pi^{k-1} : \mu(k) \neq k\right\} & \text{if } \exists \mu \in \mathcal{E}_\pi^{k-1} \text{ s.t. } \mu(k) \neq k \\ \mathcal{E}_\pi^{k-1} & \text{otherwise} \end{cases}.$$

Each matching in $\mathcal{E}_\pi^n$ is referred as a **priority matching** and they all match the same set of incompatible pairs. By construction each matching in $\mathcal{E}_\pi^n$ is Pareto-efficient. Observe that by Proposition 1 there is no trade-off between priority allocation and the number of transplants that can be arranged. In our model, all patients are indifferent between any two matchings in $\mathcal{E}_\pi^n$ and hence the priority mechanism can pick any one of them. Nevertheless there can be other considerations affecting this selection such as minimizing the number of compatible pairs that are matched.

4 The Structure of Pareto-Efficient Matchings

For any problem (N, R), partition the set of pairs $N = N_I \cup N_C$ as $\{U(N,R), O(N,R), P(N,R)\}$ where

$$\begin{aligned} U(N,R) &= \{x \in N_I : \exists \mu \in \mathcal{E}(N,R) \text{ s.t. } \mu(x) = x\}, \\ O(N,R) &= \{x \in N \setminus U(N,R) : \exists y \in U(N,R) \text{ s.t. } r_{y,x} = 1\}, \\ P(N,R) &= N \setminus (U(N,R) \cup O(N,R)). \end{aligned}$$

That is, $U(N,R)$ is the set of incompatible pairs each of which remains unmatched at a Pareto-efficient matching. We refer $U(N,R)$ as **the set of underdemanded pairs**. Set $O(N,R)$ is the set of pairs that are not underdemanded and have a mutually compatible underdemanded pair. We will refer to $O(N,R)$ as **the set of overdemanded pairs**. Set $P(N,R)$ is the remaining set of pairs, and we will refer to it as **the set of perfectly matched pairs**. Proposition 4, which we will shortly state, will justify this terminology. But first we present a simple comparative static exercise which states that if a pair is not underdemanded for a problem, then it cannot become underdemanded when a compatible pair is added to the problem.[5] So pairs that always receive a transplant at each Pareto efficient matching still do so if a compatible pair arrives.

Proposition 3 *Let (N,R) and (N^*,R^*) be such that $N^* = N \cup \{c\}$ where c is a compatible pair and $R^*_N = R$. Let $i \in N_I = N^*_I$ be such that $i \notin U(N,R)$. Then $i \notin U(N^*,R^*)$ either.*

Proof of Proposition 3. Let (N,R) and (N^*,R^*) be such that $N^* = N \cup \{c\}$ where c is a compatible pair and $R^*_N = R$. Let $i \in N_I$ be such that $i \notin U(N,R)$.

Continue the proof here.

For any $K \subset N$ let $R_K = [r_{x,y}]_{x,y \in K}$ be the **feasible exchange submatrix** for the pairs in K. We refer to (K,R_K) as a **subproblem** of (N,R). A subproblem (K,R_K) is **connected** if for any $x,y \in K$ there exist $x^1,x^2,..x^m \in K$ with $x^1 = x$ and $x^m = y$ such that for all $\ell \in \{1,...,m-1\}$, $r_{x^\ell,x^{\ell+1}} = 1$. A connected subproblem (K,R_K) is a **component** of (N,R) if there is no other connected subproblem (L,R_L) such that $K \subsetneqq L$.

Consider the subproblem $\left(N \setminus O(N,R), R_{N\setminus O(N,R)}\right)$ obtained by removal of all pairs in $O(N,R)$.

We refer to a component (K,R_K) of $\left(N \setminus O(N,R), R_{N\setminus O(N,R)}\right)$ as a **dependent component** if $K \subseteq N_I \setminus O(N,R)$ and $|K|$ is odd. We refer to a component (K,R_K) of $\left(N \setminus O(N,R), R_{N\setminus O(N,R)}\right)$ as a **self-sufficient component** if $K \cap N_C \neq \varnothing$ or $|K|$ is even. We will justify these names in the proposition given below. Let $\mathcal{D}$ denote the set of dependent components. Let $\mathcal{S}$ denote the set of self-sufficient components.

The following result characterizes the structure of the set of Pareto-efficient matchings for problem (N,R).

Proposition 4 *Given a problem (N,R), let (K,R_K) be the subproblem with $K = N \setminus O(N,R)$ (i.e. the subproblem where all overdemanded pairs are removed) and let μ be a Pareto-efficient matching for the original problem (N,R). Then,*

1. *For any pair $x \in O(N,R)$, $\mu(x) \in U(N,R)$.*

[5] It is easy to see that the same statement may not hold if an incompatible pair is added.

2.

(a) *For any self-sufficient component* (L, R_L) *of* (K, R_K)*,* $L \subseteq P(N, R)$*, and*

(b) *for any incompatible pair* $i \in L \cap N_I$*,* $\mu(i) \in L \setminus \{i\}$*.*

3.

(a) *For any dependent component* (J, R_J) *of* (K, R_K)*,* $J \subseteq U(N, R)$*, and for any pair* $i \in J$*, it is possible to match all remaining pairs in* J *with each other.*

(b) *Moreover, for any dependent component* (J, R_J) *or* (K, R_K)*, either*

   i. *one and only one pair* $i \in J$ *is matched with a pair in* $O(N, R)$ *in the Pareto-efficient matching* μ*, whereas all remaining pairs in* J *are matched with each other (so that all pairs in* J *are matched), or*

   ii. *one pair* $i \in J$ *remains unmatched in the Pareto-efficient matching* μ*, whereas all remaining pairs in* J *are matched with each other (so that only* i *remains unmatched among pairs in* J*).*

Proof of Proposition 4. We will prove the proposition by induction on the number s of compatible pairs in N.

When $s = 0$, i.e. when there are no compatible pairs, then the statement of the proposition holds by the Gallai (1963, 1964)-Edmonds (1965) Decomposition (GED) Lemma.[6]

In the inductive step, let the statement of the proposition hold when there are $s \geq 0$ compatible pairs in N. We will show that this implies that the statement holds for any problem (N, R) where there are $s + 1$ compatible pairs.

Fix any problem (N, R) where there are $s + 1$ compatible pairs in N. Pick any one of them $c \in N_C \subseteq N$. Let $(N', R') = \left(N \setminus \{c\}, R_{N\setminus\{c\}}\right)$. By the inductive assumption, Proposition 3 holds for problem (N', R').

We have two cases to consider.

Case 1. There is no pair $i \in U(N', R')$ s.t. $r_{i,c} = 1$ (i.e., pair c is not mutually compatible with any underdemanded pair of (N', R')).

[6] See Lovasz and Plummer (1986); Korte and Vygen (2002) or Roth, Sönmez, and Ünver (2005b) for a complete statement of the lemma.

Let $\nu \in \mathcal{E}(N', R')$. Since pair c is mutually compatible with only pairs in $O(N', R') \cup P(N', R') \subseteq T^{\nu}$, then no additional pair can be matched in (N, R). Thus, $\nu \in \mathcal{E}(N', R')$. This together with Proposition 3 imply that $U(N, R) = U(N', R')$. Hence, there is no pair $i \in U(N, R)$ s.t. $r_{i,c} = 1$ and therefore, $O(N, R) = O(N', R')$. Thus $P(N, R) = P(N', R') \cup \{c\}$.

Let (K, R_K) be the subproblem of (N, R) with $K = N \backslash O(N, R)$ and $(K', R'_{K'})$ be the subproblem of (N', R') with $K' = N \setminus O(N', R')$. Since $O(N, R) = O(N', R')$, $K = K' \cup \{c\}$. Recall that by inductive assumption the proposition holds for (N', R'), and therefore for any dependent component (J, R'_J) of $(K', R'_{K'})$, $J \subseteq U(N', R')$. Hence, there is no $j \in J$ s.t. $r_{j,c} = 1$. Therefore, any dependent component (J, R'_J) in $(K', R'_{K'})$ remains a dependent component of (K, R_K). Moreover, since $K \setminus K' = \{c\} \subseteq N_C$, there cannot be any additional dependent components of (K, R_K). Hence, $\mathcal{D}(K, R_K) = \mathcal{D}(K', R'_{K'})$.

By the inductive assumption since the proposition is true for (N', R'), it holds for (N, R) as well.

Case 2: There is $i \in U(N', R')$ s.t. $r_{i,c} = 1$ (i.e., pair c is mutually compatible with at least one underdemanded pair of (N', R')).

Since by the inductive assumption the proposition holds for (N', R'), $|\mathcal{D}(K', R'_{K'})| - |O(N', R')|$ incompatible pairs remain unmatched at any Pareto-efficient matching in (N', R'). We prove Case 2 through the following claims:

Claim 1: Under (N, R), $|\mathcal{D}(K', R'_{K'})| - |O(N', R')| - 1$ incompatible pairs remain unmatched at any Pareto-efficient matching.

Proof of Claim 1: Since $i \in U(N', R')$, there is a matching $\mu \in \mathcal{E}(N', R')$, with $\mu(i) = i$. Since $r_{i,c} = 1$, $\nu = \mu \cup \{(i, c)\}$ is a feasible matching under (N, R) and it leaves $|\mathcal{D}(K', R'_{K'})| - |O(N', R')| - 1$ incompatible pairs unmatched. Suppose that $\nu \notin \mathcal{E}(N, R)$, then there exists a matching η under (N, R) such that η leaves less incompatible pairs unmatched. Hence $\eta \setminus \{(\mu(c), c)\}$ is a feasible matching under (N', R'). However, this matching leaves at least $|\mathcal{D}(K', R'_{K'})| - |O(N', R')| - 1$ incompatible pairs unmatched, contradicting the fact that all Pareto-efficient matchings of (N', R') should leave the same number of incompatible pairs unmatched. Thus, all Pareto-efficient matchings of (N', R') leave exactly $|\mathcal{D}(K', R'_{K'})| - |O(N', R')| - 1$ incompatible pairs unmatched. $\Diamond$

Claim 2: Let $\nu \in \mathcal{E}(N, R)$. Then, $\nu^{-c} = \nu \setminus \{(c, \nu(c))\} \in \mathcal{E}(N', R')$.

Proof of Claim 2: Let $\nu \in \mathcal{E}(N,R)$ and $\nu^{-c} = \nu \setminus \{(c, \nu(c))\}$. By Claim 1, ν^{-c} leaves at most $|\mathcal{D}(K', R'_{K'})| - |O(N', R')|$ incompatible pairs unmatched. Since by the inductive assumption the proposition holds for (N', R'), any Pareto-efficient matching of (N', R') leaves exactly $|\mathcal{D}(K', R'_{K'})| - |O(N', R')|$ incompatible pairs unmatched. Thus $\nu^{-c} \in \mathcal{E}(N', R')$. $\Diamond$

Claim 3: If $(J, R'_J) \in \mathcal{D}(K', R'_{K'})$ such that $J \cap U(N,R) \neq \varnothing$ then $J \subseteq U(N,R)$.

Proof of Claim 3: Pick any dependent component (J, R'_J) of $(K', R'_{K'})$. Since by the inductive assumption the proposition holds for (N', R'), $J \subseteq U(N', R')$. Suppose $j \in J \cap U(N,R)$. Then there is a Pareto-efficient matching ν under (N,R) such that $\nu(j) = j$. Thus, by Claim 1, $|\mathcal{D}(K', R'_{K'})| - |O(N', R')| - 1$ incompatible pairs are unmatched in ν. Consider the matching ν^{-c} constructed from ν by removing $(\nu(c), c)$, i.e. $\nu^{-c} = \nu \setminus \{(\nu(c), c)\}$. By Claim 2, $\nu^{-c} \in \mathcal{E}(N', R')$. Since $\nu^{-c}(j) = j$, and since by the inductive assumption the proposition holds for (N', R'), then Part (3b) implies that all pairs in $J \setminus \{j\}$ are matched with each other in ν^{-c}. By Part (3a), for any $x \in J$, one can leave x unmatched, match every pair in $J \setminus \{x\}$ with each other, and keep the rest of the matching ν^{-c} the same. Let μ^{-x} be this matching. Then $\mu^{-x} \cup \{(\nu(c), c)\}$ is a Pareto-efficient matching under (N,R), and it leaves x unmatched. Hence $x \in U(N,R)$. $\Diamond$

Claim 4: Part (1) of Proposition 4 holds for (N,R).

Proof of Claim 4: Let $x \in O(N,R)$ and ν be an arbitrary Pareto-efficient matching of (N,R). We will show that $\nu(x) \in U(N,R)$.

Let $j \in U(N,R)$ s.t. $r_{x,j} = 1$ and $\mu \in \mathcal{E}(N,R)$ s.t. $\mu(j) = j$.

If $\nu(j) = j$, then the matching obtained by matching x with j instead of $\nu(x)$ and keeping all the remaining matches as in ν is Pareto-efficient. Observe that $\nu(x) \notin N_C$, since otherwise the new matching matches one more incompatible pair, j, than ν, contradicting ν is Pareto efficient. Since $\nu(x)$ is unmatched under the new Pareto-efficient matching, then $\nu(x) \in U(N,R)$ and we are done.

So assume that $\nu(j) \neq j$. We obtain a sequence of pairs $\{a_0, a_1, ..., a_k\} \subseteq N \setminus \{x\}$ as follows:

$$\begin{aligned} a_0 &= j, \\ a_1 &= \nu(a_0), \\ a_2 &= \mu(a_1), \\ a_3 &= \nu(a_2), \\ &\vdots \\ a_k &= \begin{cases} \nu(a_{k-1}) & \text{if } k \text{ is odd} \\ \mu(a_{k-1}) & \text{if } k \text{ is even} \end{cases} \end{aligned}$$

and where the last element of the sequence, a_k for $k > 0$, is (i) unmatched either in ν or in μ, or (ii) matched with x either in ν or in μ. Observe that by construction the incompatible pair $a_0 = j$ is unmatched in μ but not in ν, whereas $a_1, \ldots, a_{k-1}$ are all matched in both μ and ν. Also observe that $(a_\ell, a_{\ell+1})$ is a feasible exchange for any $\ell \in \{0, 1, \ldots, k-1\}$. Two cases are possible:

Case a. k is odd: We construct the following matching:

$$\mu' = (\mu \setminus \{(a_1, a_2), ..., (a_{k-2}, a_{k-1})\}) \cup \{(a_0, a_1), (a_2, a_3), ..., (a_{k-1}, a_k)\}.$$

Now, a_k is either (i) unmatched in μ but not in ν or (ii) matched with x in μ but not in ν. If case (i) holds, since μ' matches at least incompatible pair $a_0 = j$ in addition to all the incompatible pairs that μ matches, μ is Pareto-dominated in (N, R), a contradiction. Thus, case (ii) holds. If $x \in N_I$, then μ' is Pareto-efficient in (N, R) and $\mu'(x) = x$, contradicting x is overdemanded. If $x \in N_C$, since μ' matches at least incompatible pair $a_0 = j$ in addition to all the incompatible pairs that μ matches, μ is Pareto-dominated in (N, R), a contradiction. But then case (ii) cannot hold, either. We showed that k cannot be odd.

Case b. k is even: Then a_k is either (i) unmatched in ν but not in μ, or (ii) matched with x in ν but not in μ. We have two subcases:

case (i) holds: If $a_k \in N_C$, then we form the following matching:

$$\mu' = (\mu \setminus \{(a_1, a_2), ..., (a_{k-1}, a_k)\}) \cup \{(a_0, a_1), (a_2, a_3), ..., (a_{k-2}, a_{k-1})\}.$$

Since μ' matches $a_0 = j$ in addition to all the incompatible pairs that μ matches, μ is Pareto-dominated in (N, R), a contradiction.

Hence, $a_k \in N_I$. We construct the following matching:

$$\nu' = (\nu \setminus \{(\nu(x), x), (a_0, a_1), ..., (a_{k-2}, a_{k-1})\}) \cup \{(x, a_0), (a_1, a_2), ..., (a_{k-1}, a_k)\}.$$

Since $a_0 = j$ is mutually compatible with x, this is a feasible matching. If $\nu(x) \in N_C \cup \{x\}$, then ν' matches at least one more incompatible pair, a_k, than ν, contradicting ν is Pareto-efficient. Hence, $\nu(x) \in N_I$ and ν' matches the same number of incompatible pairs as ν. In particular, it matches all incompatible pairs matched in ν except pair $\nu(x)$, but instead it matches pair a_k. Hence, ν' is Pareto-efficient in (N, R). Since, $\nu(x)$ is unmatched in ν', we have $\nu(x) \in U(N, R)$.

case (ii) holds: Then $\nu(x) = a_k$. We form the following matching:

$$\mu' = (\mu \setminus \{(a_1, a_2), ..., (a_{k-1}, a_k)\}) \cup \{(a_0, a_1), (a_2, a_3), ..., (a_{k-2}, a_{k-1})\}.$$

If $a_k \in N_C$, then μ' matches one more incompatible pair, $a_0 = j$, than μ, contradicting that μ is Pareto-efficient. Hence, $a_k \in N_I$ and μ' matches the same number of incompatible pairs as μ. Hence, $\mu' \in \mathcal{E}(N, R)$. Since $a_k = \nu(x)$ is unmatched in μ', $\nu(x) \in U(N, R)$.$\Diamond$

Claim 5: $\mathcal{D}(K, R_K) \subseteq \mathcal{D}(K', R'_{K'})$, and $U(N, R) = \cup_{(J, R_J) \in \mathcal{D}(K, R_K)} J$.

*Proof of Claim 5:*Pick a dependent component (J, R_J) of (K, R_K).

We first show that $J \cap U(N, R) \neq \varnothing$. Let $\nu \in \mathcal{E}(N, R)$. By Claim 4, all overdemanded pairs in $O(N, R)$ are matched with underdemanded pairs in (N, R) in ν. Thus if for some $j \in J$, $\nu(j) \in O(N, R)$, then $j \in U(N, R)$, and we are done. Suppose no pair in J is matched with an overdemanded pair in ν. Since $|J|$ is odd, J consists of only incompatible pairs, and pairs in J can be mutually compatible outside J with only overdemanded pairs, then at least one pair in J is unmatched in ν. This shows that $J \cap U(N, R) \neq \varnothing$.

Since by Proposition 3 $U(N, R) \subseteq U(N', R')$, then $J \cap U(N', R') \neq \varnothing$. By the inductive assumption we have $U(N', R') = \cup_{(J', R'_{J'}) \in \mathcal{D}(K', R'_{K'})} J'$. Moreover, by Claim 3 whenever one pair in a dependent component of $(K', R'_{K'})$ is underdemanded under (N, R), so are the pairs in the whole component. Last two statements and $J \cap U(N,R) \neq \varnothing$ imply that (J, R_J) should include one or more dependent components of $(K', R_{K'})$.

We next prove that (J, R_J) should consist of only a single dependent component of $(K', R'_{K'})$: Since (J, R_J) is connected, several dependent components of $(K', R'_{K'})$ can form (J, R_J) only if some overdemanded pairs of (N', R') or c are also in J.

- Since $c \in N_C$ and (J, R_J) is a dependent component in (K, R_K), then $c \notin J$.
- We next show that $J \cap O(N', R') = \varnothing$ as follows:

  First we show that none of the overdemanded pairs of (N', R') is in $U(N,R)$: Suppose not. Pick $x \in O(N', R') \cap U(N,R)$. Let $\mu \in \mathcal{E}(N,R)$ such that $\mu(x) = x$. Then by Claim 2, $\mu^{-c} = \mu \setminus \{(c, \mu(c))\} \in \mathcal{E}(N', R')$. But then $\mu^{-c}(x) = x$, contradicting $x \in O(N', R')$.

  Suppose that $J \cap O(N', R') \neq \varnothing$. Since (J, R_J) is obtained after pairs in $O(N,R)$ are removed and $O(N', R') \cap U(N,R) = \varnothing$, then $J \cap O(N', R') \subseteq P(N,R)$. Since (J, R_J) is connected, $J \cap O(N,R) = \varnothing$, and perfectly matched pairs are not mutually compatible with any underdemanded pair, we have $J \subseteq P(N,R)$, contradicting $J \cap U(N,R) \neq \varnothing$. Hence $J \cap O(N', R') = \varnothing$.

Since none of these above two cases can happen, (J, R_J) should be a single dependent component of $(K', R'_{K'})$ implying that $(J, R_J) \in \mathcal{D}(K', R'_{K'})$.

We showed that $\mathcal{D}(K, R_K) \subseteq \mathcal{D}(K', R'_{K'})$. This together with Claim 3 and the inductive assumption imply that $U(N,R) = \cup_{(J,R_J) \in \mathcal{D}(K,R_K)} J$. $\diamondsuit$

Claim 6: Part (3) of Proposition 4 holds for (N,R).

Proof of Claim 6: Let $(J, R_J) \in \mathcal{D}(K, R_K)$. Since by Claim 5 $(J, R_J) \in \mathcal{D}(K', R'_{K'})$, then by the inductive assumption Part (3a) of the proposition holds for (J, R_J), that is, for any $j \in J$, we can match each pair in $J \setminus \{j\}$ with another pair in $J \setminus \{j\}$.

Let $\nu \in \mathcal{E}(N,R)$. By Claim 2, $\nu^{-c} = \nu \setminus \{(c, \nu(c))\} \in \mathcal{E}(N', R')$. Since by the inductive assumption Part (3b) of the proposition holds for (N', R') and ν^{-c}, either exactly one pair $j \in J$ remains unmatched or is matched with an agent $x \in O(N', R')$, and all other pairs in J are matched with another agent in $J \setminus \{j\}$ in ν^{-c}. We have two cases:

Case i. $\nu^{-c}(j) \in O(N', R')$. Then $r_{\nu(j),j} = 1$. Since $O(N', R') \cap U(N,R) = \varnothing$, $\nu(j) \notin U(N,R)$, implying in turn with $j \in U(N,R)$ that $\nu(j) \in O(N,R)$.

Case ii. $\nu^{-c}(j) = j$: Then $\nu(j) \in \{c, j\}$. Consider the case $\nu(j) = c$. We have $r_{j,c} = 1$. Since $c \in N_C$, $c \notin U(N, R)$, implying in turn with $j \in U(N, R)$ that $\nu(j) = c \in O(N, R)$.

Hence, $\nu(j) \in O(N, R) \cup \{j\}$, and Part (3b) of the proposition holds for (N, R). ◊

Claims 4, 5 and 6 immediately imply that Part (2) of Proposition 4 holds for (N, R). ◊

5 Concluding Comments

Although we talked about the structure of the Pareto-efficient matchings, we did not propose a computationally feasible method to find one. It turns out that an existent algorithm can be tailored to find one. Edmonds (1965) proposed also another algorithm that can be used to find a maximal *edge-weighted* matching. Economic translation of this problem conveys a two-way exchange environment in which each mutually compatible pairs' exchange is weighted by a number. This algorithm finds a matching that maximizes the sum of weights' of conducted exchanges in polynomial time. If we weight the exchanges between a compatible pair and an incompatible pair by 1 and those between incompatible pairs by 2, then we can use Edmonds' weighted matching algorithm to find a Pareto-efficient matching. Moreover, when we would like to minimize the number of matched compatible pairs, we can increase the weights between mutually compatible incompatible pairs to $2+\varepsilon$ for some sufficiently small and positive ε.

Our model also extends to another new paradigm in kidney exchange: exchanges initiated by *non-directed donors (NDD)*, i.e., donors who would like to donate one of their kidneys to a complete stranger. The number of such donors has steadily been high in the last 5 years, about 100 donors per year in the US. Montgomery et al. (2006) reported exchange chains of incompatible pairs $i_1, i_2, ..., i_n$ initiated through NDDs, in which an NDD donates to i_1's patient, i_1's donor donates to i_2's patient,...,i_{n-1}'s donor donates to i_n'patient, and finally, i_n's donor donates to a high-priority patient waiting in the deceased-donor list. Roth et al. (2006) proposed that these transplants in such a chain need not be simultaneous unlike in *paired kidney exchanges*. If we start the transplantation operations in the chain with the NDD end, then, even if one donor reneges, the patients who are yet to receive a transplant can participate in other exchanges with their paired-donors, who have not yet donated. Rees et al. (2009) reported a series of non-simultaneous *continuing* NDD-chain transplants, in which the donor of the end-pair of the chain plays the role of the NDD in the next batch of the exchange run. In our model, If we treat NDDs as compatible pairs and are allowed to create one pair-long NDD-chains, all our results in this paper will continue to hold in this domain.

In this paper, we stated a framework and properties of matchings for kidney exchanges including compatible pairs. Our future research agenda includes designing mechanisms for this framework, such as the priority mechanism we stated, and possibly implementing those in the field.

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